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MATHEMATICAL ASPECTS OF THE OFF-LINE PROGRAMMING OF FILAMENT WINDING MACHINES FOR GENERAL SURFACES OF REVOLUTION

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This report contains a reasonably complete description of the analytical, geometrical, numerical, and computational aspects of the off-line programming of a rotating-traversing composite filament winding machine for wrapping general axisymmetric surfaces of revolution. The topics addressed are geodesic winding paths, path/angle relations, quasi-geodesic paths, wrappable paths and surfaces, path behavior near turning points, a square root singularity quadrature formula, piecewise linear approximation, path

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INTRODUCTION

This report documents the mathematical details of a solution to the following problem: given a general surface of revolution, produce precision information off-line that a rotating-traversing (2-axis) filament winding machine needs in order to wrap (cover) the surface uniformly.

Off-line programming refers to the use of analytical and computational methods instead of manual methods for teaching the machine the motions it needs to perform in order to uniformly cover a surface with wrapped filament. Off-line programming is important because winding data can be produced almost immediately upon specification of the surface profile function. Also, the filament winding angle can be tailored to some extent and various different paths may be tested on various different surfaces without requiring the production of the physical surface.

On-line programming, on the other hand, almost completely avoids any need for mathematical analysis, but it does so at a price. Manual teaching is time consuming, tedious, error prone, less flexible, and it removes a machine from production while it is being manually put through its paces. Also, the very first requirement of manual teaching is the production of the physical surface or mandrel.

Various important mathematical areas involved in this report are differential geometry, variational calculus, asymptotic expansions, numerical quadrature, error analysis, finite differences, numerical approximation, number theory, and of course, computer programming.

GEODESIC WINDING PATHS

The curve on the surface along which we wish to wind a composite filament will be referred to as a winding path. The ideal winding path is a geodesic

because the filament will have absolutely no tendency to slip subsequent to its meeting with the surface. If there is no friction between filament and surface, a purely geodesic winding path is mandatory. If there is considerable friction between filament and surface, one may be able to deviate considerably from a geodesic winding path with no ill effects. In any case, the geodesic is the logical starting point.

A good operational definition of a geodesic path is simply the path of least length between two (nearby) points on the surface. A geodesic in the plane is a straight line. A geodesic on a cylinder is a helix. A geodesic on a sphere is a great circle.

A surface of revolution is defined by its profile or radius function $r(x)$, where x measures the distance along the axis of the surface.

Using a right-handed (x,y,z) coordinate system, the equation of the axisymmetric surface of revolution is given by

$$y^2 + z^2 = r(x)^2$$

For any path on the surface (not necessarily geodesic),

$$y = r(x) \sin \theta$$

$$z = r(x) \cos \theta$$

where θ is the angular amount of wrap of the path around the axis of the surface, beginning at the point $(0,0,r(0))$ $(\theta = 0)$.

The differential of arc length for any point on the path is given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

but

$$dy = dr \sin \theta + r \cos \theta \, d\theta$$

and

$$dz = dr \cos\theta - r \sin\theta \, d\theta$$

Therefore

$$(ds)^2 = (dx)^2 + (dr \sin\theta + r \cos\theta \, d\theta)^2 + (dr \cos\theta - r \sin\theta \, d\theta)^2$$

Squaring and simplifying, we have

$$(ds)^2 = (dx)^2 + (dr)^2 + r^2(d\theta)^2$$

or

$$(ds)^2 = (dx)^2 + \left(\frac{dr}{dx}\right)^2(dx)^2 + r^2\left(\frac{d\theta}{dx}\right)^2(dx)^2 = (1+r'^2+r^2\theta'^2)(dx)^2$$

or

$$ds = (1+r'^2+r^2\theta'^2)^{1/2} \, dx$$

Therefore, in terms of the behavior of r and θ as functions of x, we may calculate the length of any path on the x interval (a,b) by

$$s = \int_a^b (1+r'^2+r^2\theta'^2)^{1/2} \, dx$$

We will not actually have occasion to calculate s, but this integral for s is important in obtaining the θ behavior of a geodesic path on the surface.

The fundamental problem of the calculus of variations is to find $\theta(x)$ which minimizes the integral

$$\int_a^b F(x,\theta,\theta')dx$$

This minimization problem is equivalent to solving the Euler differential equation

$$\frac{\partial F}{\partial \theta} = \frac{d}{dx} \frac{\partial F}{\partial \theta'}$$

but in our case,

$$F(x,\theta,\theta') = (1+r'^2+r^2\theta'^2)^{1/2}$$

and this expression is not dependent on θ. Therefore

$$\frac{\partial F}{\partial \theta} = 0 = \frac{d}{dx}\,\frac{\partial F}{\partial \theta'}$$

and

$$\frac{\partial F}{\partial \theta'} = c = \text{constant}$$

Now

$$\frac{\partial F}{\partial \theta'} = r^2\theta'(1+r'^2+r^2\theta'^2)^{-\frac{1}{2}} = r^2\left(\frac{1+r'^2}{\theta'^2} + r^2\right)^{-\frac{1}{2}} = c$$

Geodesic curves which pass through the axis exactly twice (at the poles) will be called meridians. Nongeodesic curves having a constant x coordinate will be referred to as parallels or hoops.

Assume the geodesic is tangent to a parallel at some point x_o (the usual starting point will be $x_o = 0$). Such a point represents a stationary point of x as a function of θ. Therefore

$$\frac{dx}{d\theta} = 0 \text{ at } x = x_o$$

or

$$\theta' = \infty \text{ at } x = x_o$$

We may therefore conclude from

$$r^2\left(\frac{1+r'^2}{\theta'^2} + r^2\right)^{-\frac{1}{2}} = c$$

that

$$c = r(x_o) = r_o$$

Therefore

$$r^2\left(\frac{1+r'^2}{\theta'^2} + r^2\right)^{-\frac{1}{2}} = r_o$$

The point x_0 will be called a turning point of the geodesic and the corresponding radius r_0 will be called the polar radius. The turning points of the geodesic represent the boundaries between the wrapped region of the surface and the unwrapped regions of the surface. Given a surface of revolution, we are theoretically at liberty to specify any point we wish as a turning point of a geodesic; having done this, we have determined the entire geodesic and the other turning point as well. These facts imply that in order to wrap or wind along a pure geodesic path, we must wind between two precisely equal polar radii and $r(x)$ may not drop below the polar radius r_0 in the wrapped region. This situation is rather restrictive, and we will discuss how to overcome it later.

Solving

$$r^2\left(\frac{1+r'^2}{\theta'^2} + r^2\right)^{-1/2} = r_0$$

for θ', we have

$$\theta'(x) = \frac{r_0}{r}\left(\frac{1+r'^2}{r^2-r_0^2}\right)^{1/2}$$

Therefore, θ for a geodesic with polar radius r_0 is given by

$$\theta(x) = \int_0^x \frac{r_0}{r(t)}\left(\frac{1+r'(t)^2}{r(t)^2-r_0^2}\right)^{1/2} dt$$

This integral is improper, of course, because the integrand becomes infinite at the turning points. Intuition assures us, however, that the integral does indeed exist and we will subsequently develop an asymptotic approximation for it for small x. $\theta(x)$ is, in fact, proportional to $\sqrt{x}$ for small x.

PATH-ANGLE RELATIONS

Let the hoop angle h be defined as the angle between a hoop and the path. The angle between the path and a meridian will be called the winding angle w.

More precisely, the hoop angle h is the angle between a vector tangent to the path and a vector tangent to the hoop.

Letting i, j, and k be the usual coordinate basis vectors, the unit tangent vector to any curve on the surface is given by

$$T_p = (idx + jdy + kdz)/ds$$

$$= (idx + j(dr \sin\theta + r\cos\theta\, d\theta) + k(dr\cos\theta - r\sin\theta\, d\theta))/ds$$

for the particular case of a hoop or parallel, $dx = 0 = dr$, therefore

$$T_h = (jr\cos\theta\, d\theta - kr\sin\theta\, d\theta)/ds$$

For a hoop we also have $ds = rd\theta$, therefore,

$$T_h = j\cos\theta - k\sin\theta$$

Taking the dot or inner product of these two unit tangent vectors, we have

$$\cos h = T_p \cdot T_h$$

$$= \{(dr\sin\theta + r\cos\theta\, d\theta)\cos\theta - (dr\cos\theta - r\sin\theta\, d\theta)\sin\theta\}/ds$$

$$= r\frac{d\theta}{ds}$$

$$= r\theta'\frac{dx}{ds}$$

$$= r\theta'(1+r'^2+r^2\theta'^2)^{-½}$$

$$= r^2\left(\frac{1+r'^2}{\theta'^2} + r^2\right)^{-½}/r$$

Therefore

$$\cos h = \frac{r_o}{r}$$

Aside from its simplicity, the interesting thing about this relation is that it not only holds when r is not differentiable, but it also holds even when r is not continuous. This means that if x = a is a discontinuity of r, and r is

set equal to any value between r(a-o) and r(a+o), the cosine of the correspr h will still be given by r_o/r. Since meridians and parallels inter ogonally, the winding angle w is given by

$$\sin w = \frac{r_o}{r}$$

These formulas for hoop and winding angles will hold (in slightly modified form) for subsequently defined nongeodesic paths as well.

QUASI-GEODESIC PATHS

The following surface is guaranteed to be wrappable on a purely geodesic path. We are considering just one circuit here, beginning at the left-hand turning point (x=0), winding to the right-hand turning point (x=L), and returning to the left-hand turning point.

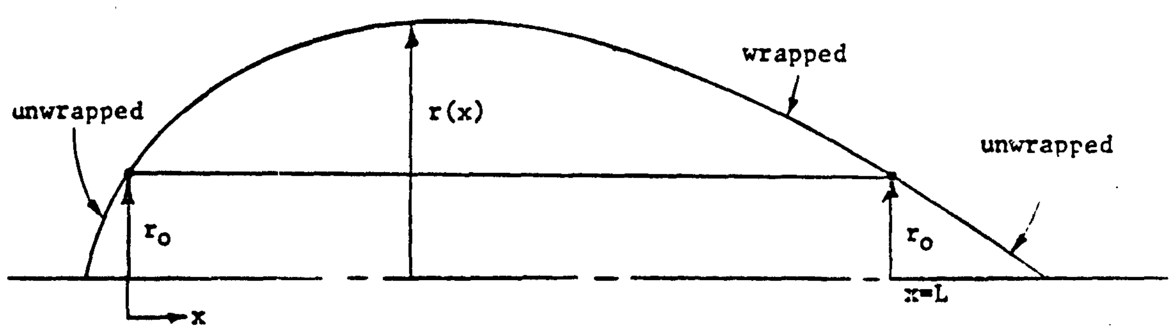

Figure 1. Equal polar radii for pure geodesic.

The reasons why this surface is guaranteed to be geodesically wrappable are

1. $r(x) = r_o$ at <u>exactly</u> two points (x=0 and x=L)
2. $r(x) > r_o$ for $0 < x < L$
3. $r''(x) \leqslant 0$ for $0 \leqslant x \leqslant L$

We will address the problem presented by violating condition #1 in this section, leaving the problem of violating condition #3 for the next section.

Suppose we wished to wrap a surface similar to the previous one, but we wanted two unequal polar radii r_{01} and r_{02} ($r_{02} > r_{01}$).

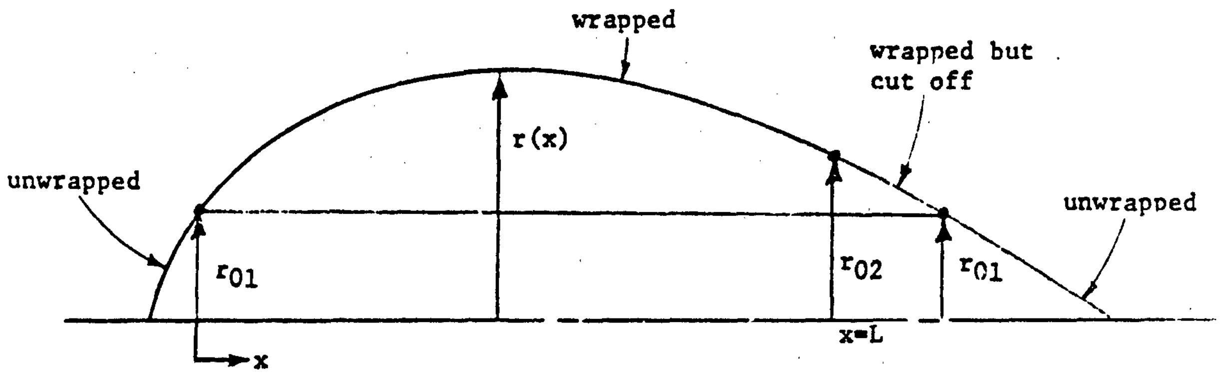

Figure 2. Overwrap for unequal polar radii.

If we want r_{01} and r_{02} to correspond to turning points, the task is mathematically impossible if we insist on a purely geodesic path. One alternative we have in this case, however, is to wind a pure geodesic farther out on the right-hand end of the surface than we want to and subsequently cut off the unwanted surface to get our larger right-hand polar radius. This alternative is not generally viable, however, because if the resulting composite surface is to be used as a pressure vessel, the cutting of the fibers or filaments may not be desirable for reasons of structural integrity.

There is yet another alternative, however, which gives us considerable flexibility in design and fabrication of the composite surface. We use the simple but elegant expedient of thinking of r_o not as merely a number, but as a function $r_o(x)$. Our quasi-geodesic path is therefore defined by

$$\theta'(x) = \frac{r_o(x)}{r(x)} \left(\frac{1+r'(x)^2}{r(x)^2 - r_o(x)^2}\right)^{\frac{1}{2}}$$

where $r(0) = r_0(0)$, $r(L) = r_0(L)$, and $r(x) > r_0(x)$ for $0 < x < L$. The invented function $r_0(x)$ will be referred to as the polar radius function.

The schematic for the previous surface could therefore look like Figure 3.

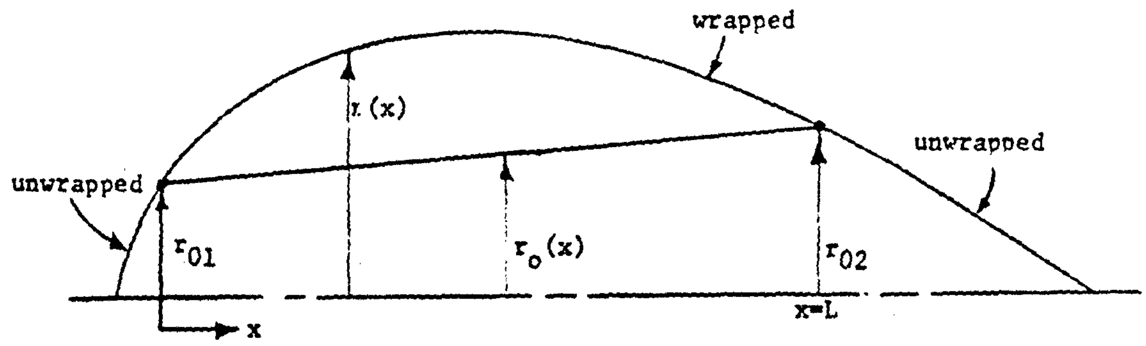

Figure 3. Quasi-geodesic with unequal polar radii.

Here, we have our turning points exactly where we want them and our polar diameters are exactly what we want them to be. By using the linearly varying polar radius function $r_0(x)$, we have given up one small thing - a pure geodesic path. Our path is now geodesic <u>nowhere</u>, but very nearly geodesic <u>everywhere</u>! By inventing the polar radius function, we have in one fell swoop generated an infinite number of similar geodesics and selected a single point from each!

We might also have defined $r_0(x)$ as shown in Figure 4.

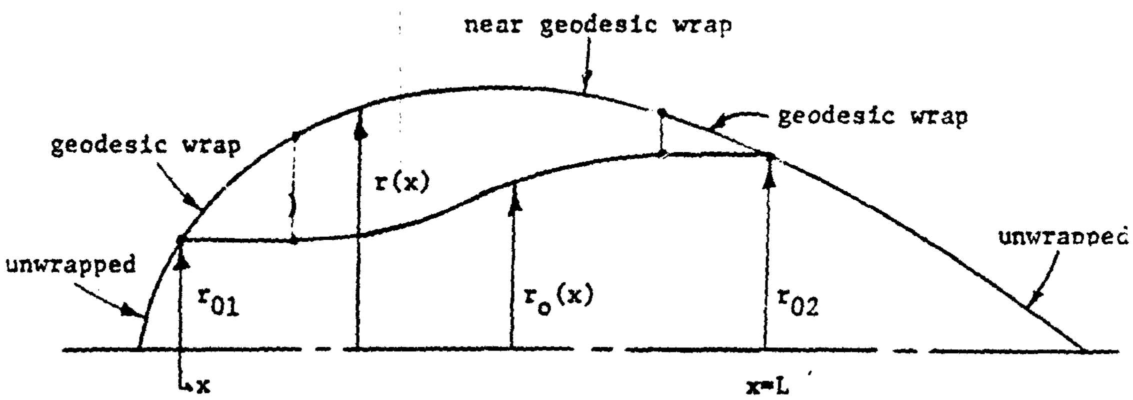

Figure 4. Partially pure geodesic with unequal polar radii.

Here, we have allowed $r_o(x)$ to be constant near the ends of the surface; whenever $r_o(x)$ is constant, a pure geodesic wrap results.

If we wanted greater hoop strength in the resulting composite surface, we could define $r_o(x)$ as shown in Figure 5.

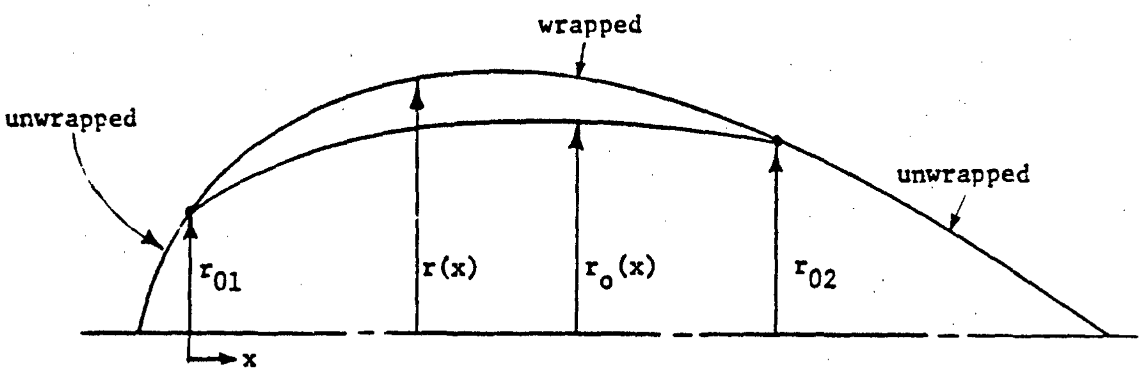

Figure 5. Polar radius function for higher hoop strength.

If we wanted greater axial stiffness, we might define $r_o(x)$ as shown in Figure 6.

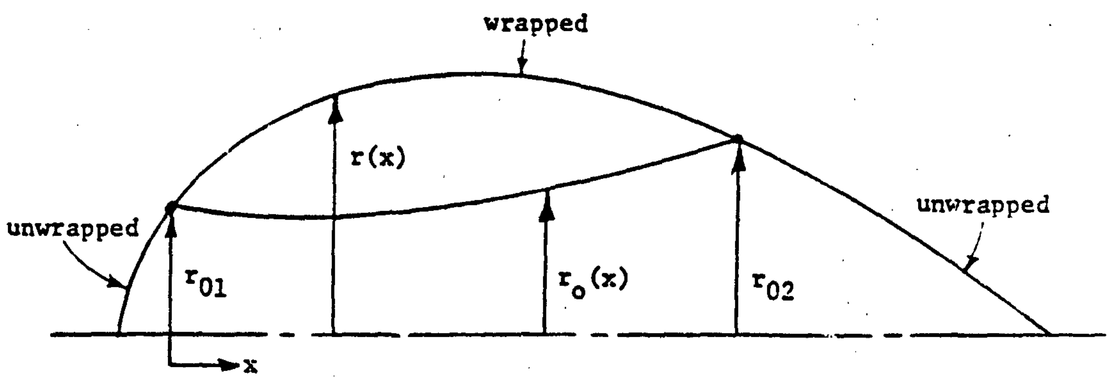

Figure 6. Polar radius function for higher longitudinal stiffness.

Needless to say, the further r_o deviates from a constant function, the further the path will deviate from a pure geodesic. Too abrupt a deviation from a geodesic path can produce slipping and catching of the filament, producing a

nonuniform surface. For a convex surface, as pictured in the previous drawings, slippage is the only problem we can have with a quasi-geodesic. For a nonconvex surface, another problem can arise - lift-off or bridging, where the filament, in some region, lifts off the surface and forms a bridge between two points on the surface.

The following surface could suffer from bridging in the region indicated:

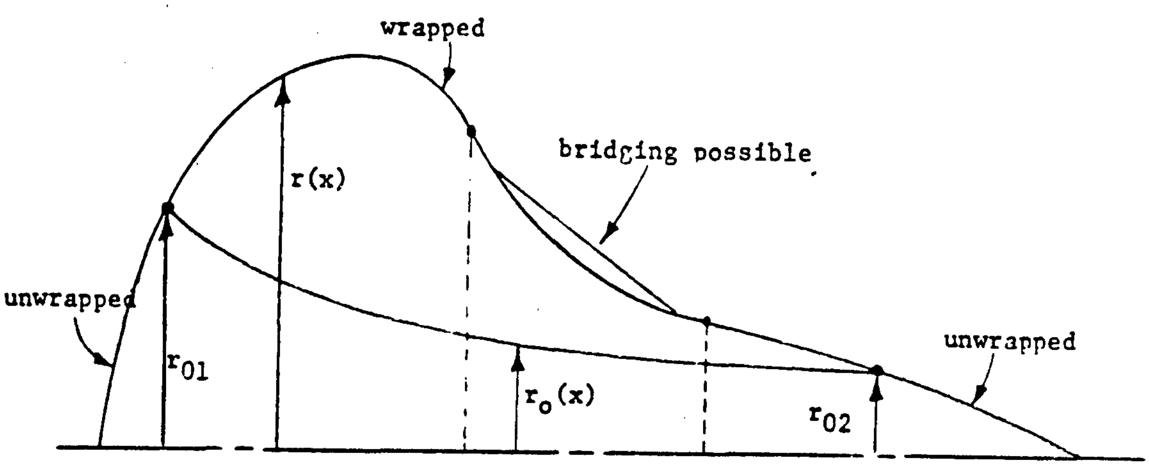

Figure 7. Nonconvex surface, possible bridging.

The avoidance of bridging will be the subject of the next section.

WRAPPABLE PATHS AND SURFACES

As mentioned in the previous section, all convex surfaces are wrappable, provided we select r_o properly and no slippage problems occur. When $r''(x) > 0$ in any region, however, we may get bridging - where the filament wants to lift off the surface and form a bridge between two points on the surface. If bridging can occur, the path, geodesic though it may be, it said to be unwrappable. In order to avoid bridging, it is first necessary to develop a mathematical condition for the occurrence of bridging.

The gradient vector to the surface

$$S(x,y,z) = y^2 + z^2 - r(x)^2 = 0$$

will always point perpendicularly outward from the surface. The path curvature vector may point either inward or outward, however. As the curvature vector K approaches the tangent plane from below, we are approaching a bridging situation. If the curvature vector subsequently points outward from the tangent plane at any point, we have a definite bridging situation. The condition which must therefore hold in order to avoid bridging is that the inner or dot product of the gradient vector and the curvature vector must remain negative.

A vector proportional to the gradient of surface S is given by

$$G = - irr' + jy + kz$$

If $P = ix + jy + kz$ is a point on the winding path, the curvature vector of the path is given by

$$K = i \frac{d^2x}{ds^2} + j \frac{d^2y}{ds^2} + k \frac{d^2z}{ds^2}$$

Therefore

$$G \cdot K = - rr' \frac{d^2x}{ds^2} + y \frac{d^2y}{ds^2} + z \frac{d^2z}{ds^2}$$

and we must have

$$G \cdot K \leq 0$$

in order to avoid bridging.

For any function u(x), we have, by the chain rule:

$$\frac{d^2u}{ds^2} = (s'u''-u's'')/s'^3$$

Therefore

$$s'^3 G \cdot K = - rr'(-s'') + y(s'y''-y's'') + z(s'z''-z's'') =$$

$$(rr'-yy'-zz')s'' + (yy''+zz'')s'$$

Using $y = r \sin \theta$ and $z = r \cos \theta$, doing the differentiations and simplifying, we find that

$$rr' - yy' - zz' = 0$$

$$yy'' + zz'' = rr'' - r^2\theta'^2$$

Therefore

$$G \cdot K = r(r''-r\theta'^2)/s'^2$$

Since only the sign of this function matters, we define the bridging function to be

$$B(x) = r'' - r\theta'^2$$

where bridging takes place if $B(x) > 0$ and not if $B(x) \leqslant 0$.

While it is already intuitively obvious that a convex surface cannot experience bridging, it is now also mathematically obvious because B cannot be positive if r'' is negative. On the other hand, if the surface is concave ($r'' > 0$) in any region, B <u>can</u> be positive if θ' is not sufficiently large. In order to avoid bridging, we simply make sure that θ' will always be sufficiently large by making r_o sufficiently large.

Writing the condition for no bridging:

$$r'' - r\theta'^2 \leqslant 0$$

and inserting the expression for θ', we have

$$r'' - r \cdot \frac{r_o^2}{r^2} \cdot \frac{1+r'^2}{r^2 - r_o^2} \leqslant 0$$

Isolating r_o on one side, we have

$$r_o \geqslant \left(\frac{r^3 r''}{1+r'^2 + rr''}\right)^{\frac{1}{2}}$$

We define the right-hand side of this inequality to be the polar radius lower bound function $\Lambda(x)$

$$\Lambda(x) = \begin{cases} 0 & \text{if } r'' \leq 0 \\ \left(\dfrac{r^3 r''}{1+r'^2 + rr''}\right)^{1/2} & \text{if } r'' > 0 \end{cases}$$

We see therefore, that bridging can be avoided if we make certain that $r_o(x) \geq \Lambda(x)$ everywhere. Note that if $r'' = \infty$ (a sharp valley), then $\Lambda = r$.

PATH BEHAVIOR NEAR TURNING POINTS

In this section we obtain one-term asymptotic expressions for $\theta'(x)$ in the vicinity of turning points ($\theta' = \infty$). The general expression for $\theta'(x)$ is given by

$$\theta'(x) = \frac{r_o(x)}{r(x)} \left(\frac{1+r'(x)^2}{r^2(x) - r_o^2(x)}\right)^{1/2}$$

For x near zero we have

$$r(x) \sim r(o) + xr'(o)$$

and

$$r_o(x) \sim r_o(o) + xr_o'(o)$$

Now

$$r(x)^2 - r_o(x)^2 = (r(x)-r_o(x))(r(x)+r_o(x))$$

but

$$r_o(o) = r(o)$$

Therefore, we have

$$r(x) - r_o(x) \sim x(r'(o)-r_o'(o))$$

and

$$r(x)^2 - r_o(x)^2 \sim x(r'(o)-r_o'(o))(2r(o))$$

Ultimately,

$$\theta'(x) \sim \frac{r_0(0)}{r(0)} \left(\frac{1 + r'(0)^2}{2xr(0)(r'(0)-r_0'(0))}\right)^{1/2}$$

$$= \frac{1}{\sqrt{x}} \left(\frac{1 + r'(0)^2}{2r(0)(r'(0)-r_0'(0))}\right)^{1/2}$$

and

$$\theta(x) \sim \sqrt{x}\left(\frac{2(1+r'(0)^2)}{r(0)(r'(0)-r_0'(0))}\right)^{1/2}$$

For x near L we have

$$r(x) \sim r(L) + (x-L)r'(L)$$

and

$$r_0(x) \sim r_0(L) + (x-L)r_0'(L)$$

but

$$r_0(L) = r(L)$$

Therefore

$$r(x) - r_0(x) \sim (x-L)(r'(L)-r_0'(L))$$

and

$$r(x)^2 - r_0(x)^2 \sim (x-L)(r'(L)-r_0'(L))(2r(L))$$

$$= 2(L-x)r(L)(r_0'(L)-r'(L))$$

and ultimately,

$$\theta'(x) \sim \frac{r_0(L)}{r(L)} \left(\frac{1 + r'(L)^2}{2(L-x)r(L)(r_0'(L)-r'(L))}\right)^{1/2}$$

$$= \frac{1}{\sqrt{L-x}} \left(\frac{1 + r'(L)^2}{2r(L)(r_0'(L)-r'(L))}\right)^{1/2}$$

We see, therefore, that $\theta'(x)$ has square root singularities at the left and right turning points. Integration of square root singular functions will be discussed in the next section.

A SQUARE ROOT SINGULARITY QUADRATURE FORMULA

In this section we will develop a one-point Gaussian quadrature formula with error term for integrating square root singularity functions in the vicinity of the singularity. We have the following integral to evaluate:

$$I = \int_a^{a+h} \frac{f(t)}{\sqrt{t}}\,dt \quad \text{for } a \geq 0$$

We rewrite this integral as two separate integrals:

$$I = \int_a^x \frac{f(t)}{\sqrt{t}}\,dt - \int_{a+h}^x \frac{f(t)}{\sqrt{t}}\,dt$$

We will use the following formula for integration-by-parts on numerous occasions:

$$\int_a^b f(x)g(x)dx = f(b)\int_a^b g(x)dx - \int_a^b f'(x)\int_a^x g(t)dtdx$$

Using integration-by-parts on I, we have

$$I = f(x)\int_a^x t^{-1/2}dt - \int_a^x f'(t)\int_a^t u^{-1/2}dudt$$

$$- \left\{f(x)\int_{a+h}^x t^{-1/2}dt - \int_{a+h}^x f'(t)\int_{a+h}^t u^{-1/2}dudt\right\}$$

$$= 2f(x)(x^{1/2}-a^{1/2}) - \int_a^x 2f'(t)(t^{1/2}-a^{1/2})dt$$

$$-2f(x)(x^{1/2}-(a+h)^{1/2}) + \int_{a+h}^x 2f'(t)(t^{1/2}-(a+h)^{1/2})dt$$

$$= 2f(x)((a+h)^{1/2}-a^{1/2})$$

$$- \int_a^x 2f'(t)(t^{1/2}-a^{1/2})dt$$

$$+ \int_{a+h}^x 2f'(t)(t^{1/2}-(a+h)^{1/2})dt$$

Let

$$E = I - 2f(x)((a+h)^{1/2}-a^{1/2})$$

$$= I - \frac{2hf(x)}{(a+h)^{1/2} + a^{1/2}}$$

Therefore

$$\frac{E}{2} = - \int_a^x f'(t)(t^{1/2}-a^{1/2})dt$$

$$+ \int_{a+h}^x f'(t)(t^{1/2}-(a+h)^{1/2})dt$$

Using integration-by-parts again on the last two integrals gives us

$$\frac{E}{2} = - \{f'(x)\int_a^x t^{1/2} - a^{1/2}dt - \int_a^x f''(t)\int_a^t u^{1/2} - a^{1/2}dudt\}$$

$$+ f'(x)\int_{a+h}^x t^{1/2} - (a+h)^{1/2}dt - \int_{a+h}^x f''(t)\int_{a+h}^t u^{1/2} - (a+h)^{1/2}dudt$$

$$= - f'(x)(2/3t^{3/2}-a^{1/2}t)\Big|_a^x + \int_a^x f''(t)(2/3u^{3/2}-a^{1/2}u)\Big|_a^t dt$$

$$+ f'(x)(2/3t^{3/2}-(a+h)^{1/2}t)\Big|_{a+h}^x - \int_{a+h}^x f''(t)(2/3u^{3/2}-(a+h)^{1/2}u)\Big|_{a+h}^t dt$$

$$= -f'(x)(2/3x^{3/2}-a^{1/2}x-2/3a^{3/2}+a^{3/2})$$

$$+ \int_a^x f''(t)(2/3t^{3/2}-a^{1/2}t-2/3a^{3/2}+a^{3/2})dt$$

$$+ f'(x)(2/3x^{3/2}-(a+h)^{1/2}x-2/3(a+h)^{3/2} + (a+h)^{3/2})$$

$$+ \int_x^{a+h} f''(t)(2/3t^{3/2}-(a+h)^{1/2}t-2/3(a+h)^{3/2}+(a+h)^{3/2})dt$$

Therefore

$$\frac{E}{2} = f'(x)(x(a^{1/2}-(a+h)^{1/2}) + {}_1/_3((a+h)^{3/2}-a^{3/2}))$$

$$+ \int_a^x f''(t)({}_2/_3t^{3/2}-a^{1/2}t+{}_1/_3a^{3/2})dt$$

$$+ \int_x^{a+h} f''(t)({}_2/_3t^{3/2}-(a+h)^{1/2}t+{}_1/_3(a+h)^{3/2})dt$$

If we now pick x to zero the $f'(x)$ term, we will get a quadrature formula which is exact for linear f. Therefore, let

$$x(a^{1/2}-(a+h)^{1/2}) + {}_1/_3((a+h)^{3/2}-a^{3/2}) = 0$$

Solving for x gives

$$x = \frac{((a+h)^{3/2}-a^{3/2})}{3((a+h)^{1/2}-a^{1/2})} = \frac{((a+h)^{3/2}-a^{3/2})((a+h)^{1/2}+a^{1/2})}{3(a+h-a)}$$

$$= \frac{(a+h)^2 - a^{3/2}(a+h)^{1/2} + a^{1/2}(a+h)^{3/2}-a^2}{3h}$$

$$= \frac{2ah+h^2 + a^{1/2}(a+h)^{1/2}(a+h-a)}{3h}$$

$$= (2a+h+(a^2+ah)^{1/2})/3$$

We therefore have

$$I = \int_a^{a+h} \frac{f(t)}{\sqrt{t}}\, dt = \frac{2hf(x)}{(a+h)^{1/2}+a^{1/2}} + E$$

where

$$x = {}_1/_3(2a+h+(a^2+ah)^{1/2})$$

and

$$\frac{E}{2} = \int_a^x f''(t)({}_2/_3t^{3/2}-a^{1/2}t+{}_1/_3a^{3/2})dt$$

$$+ \int_x^{a+h} f''(t)({}_2/_3t^{3/2}-(a+h)^{1/2}t+{}_1/_3(a+h)^{3/2})dt$$

We note here that if $a = 0$, then $x = \frac{h}{3}$, and for large a (relative to h),

$$(a+h)^{1/2} = a^{1/2}(1 + \frac{h}{a})^{1/2} \sim a^{1/2}(1 + \frac{h}{2a})$$

Hence

$$x \sim 1/3(2a+h+a(1 + \frac{h}{2a}))$$

$$= 1/3(2a+h+a+ \frac{h}{2})$$

$$= 1/3(3a + \frac{3h}{2}) = a + \frac{h}{2}$$

We see, therefore, that for large a, this quadrature formula becomes the mid-point rule.

We now devote the remainder of this section to developing an upper bound on the absolute error $|E|$. From the expression for E, define:

$$\phi(t) = 2/3t^{3/2} - a^{1/2}t + 1/3a^{3/2}$$

and

$$\psi(t) = 2/3t^{3/2} - (a+h)^{1/2}t + 1/3(a+h)^{3/2}$$

We first prove that ϕ and ψ are non-negative in their respective ranges of integration. This is important when we apply the absolute value triangle inequalities to the expression for E.

$$\phi(a) = 2/3a^{3/2} - a^{3/2} + 1/3a^{3/2} = 0$$

$$\phi'(t) = t^{1/2} - a^{1/2} \quad , \quad \phi'(a) = 0$$

$$\phi''(t) = \frac{1}{2\sqrt{t}} > 0$$

Hence, ϕ and ϕ' are zero at a and ϕ is concave upward to the right of a. This shows that ϕ is positive to the right of a.

Now

$$\psi(a+h) = {}_2/_3(a+h)^{3/2} - (a+h)^{3/2} + {}_1/_3(a+h)^{3/2} = 0$$

$$\psi'(t) = t^{1/2} - (a+h)^{1/2} \quad , \quad \psi'(a+h) = 0$$

$$\psi''(t) = \frac{1}{2\sqrt{t}} > 0$$

Hence, ψ is positive to the left of a+h.

Using the absolute value triangle inequalities for sums and integrals on the expression for $\frac{E}{2}$ gives us:

$$\left|\frac{E}{2}\right| = \left|\int_a^x f''(t)\phi(t)dt + \int_x^{a+h} f''(t)\psi(t)dt\right|$$

$$\leq \left|\int_a^x f''(t)\phi(t)dt\right| + \left|\int_x^{a+h} f''(t)\psi(t)dt\right|$$

$$\leq \int_a^x |f''(t)|\phi(t)dt + \int_x^{a+h} |f''(t)|\psi(t)dt$$

$$\leq \|f''\| \left(\int_a^x \phi(t)dt + \int_x^{a+h} \psi(t)dt\right)$$

where

$$\|f''\| = \sup\{|f''(t)| : a \leq t \leq a+h\}$$

Calculating the integrals of ϕ and ψ:

$$\int_a^x \phi(t)dt = \int_a^x {}_2/_3t^{3/2} - a^{1/2}t + {}_1/_3a^{3/2}\,dt$$

$$= {}_4/_{15}(x^{5/2}-a^{5/2}) - \tfrac{1}{2}a^{1/2}(x^2-a^2) + {}_1/_3a^{3/2}(x-a)$$

$$\int_x^{a+h} \psi(t)dt = \int_x^{a+h} {}_2/_3t^{3/2} - (a+h)^{1/2}t + {}_1/_3(a+h)^{3/2}\,dt$$

$$= {}_4/_{15}((a+h)^{5/2}-x^{5/2}) - \tfrac{1}{2}(a+h)^{1/2}((a+h)^2-x^2) + {}_1/_3(a+h)^{3/2}(a+h-x)$$

We therefore have

$$\left|\frac{E}{2}\right| \leq \|f''\|\{{}_4/_{15}((a+h)^{5/2}-a^{5/2}) - \tfrac{1}{2}x^2(a^{1/2}-(a+h)^{1/2})$$

$$+ {}_1/_3x(a^{3/2}-(a+h)^{3/2}) + \tfrac{1}{2}a^{5/2} - {}_1/_3a^{5/2}$$

$$- \tfrac{1}{2}(a+h)^{5/2} + {}_1/_3(a+h)^{5/2}\}$$

Therefore

$$15\,|E| \leq \|f''\|\{8((a+h)^{5/2}-a^{5/2})$$

$$-15x^2(a^{1/2}-(a+h)^{1/2}) + 10x(a^{3/2}-(a+h)^{3/2})$$

$$+15a^{5/2} - 10a^{5/2} - 15(a+h)^{5/2} + 10(a+h)^{5/2}\}$$

$$= \|f''\|\{15x^2((a+h)^{1/2}-a^{1/2})$$

$$- 10x((a+h)^{3/2}-a^{3/2}) + 3(a+h)^{5/2}-3a^{5/2}\}$$

Recall now that

$$x = {}_1/_3(2a+h+a^{1/2}(a+h)^{1/2})$$

Now let $k = a/h$, $p = k^{1/2}$, and $q = (k+1)^{1/2}$. Therefore, $a^{1/2} = ph^{1/2}$ and $(a+h)^{1/2} = qh^{1/2}$.

Hence

$$x = {}_1/_3(2kh+h+ph^{1/2}qh^{1/2})$$

$$= \frac{h}{3}\,(2k+1+pq)$$

and

$$x^2 = \frac{h^2}{9}\,(4k^2+1+p^2q^2+4k+4kpq+2pq)$$

$$= \frac{h^2}{9}\,(4k^2+1+k(k+1)+4k+pq(4k+2))$$

$$= \frac{h^2}{9}\,(5k^2+5k+1+pq(4k+2))$$

Our error bound therefore becomes

$$15\,|E| \leq \|f''\|\{15 \cdot \frac{h^2}{9}\,(5k^2+5k+1+pq(4k+2))(qh^{1/2}-ph^{1/2})$$

$$- 10 \cdot \frac{h}{3}\,(2k+1+pq)(q^3h^{3/2}-p^3h^{3/2})$$

$$+ 3q^5h^{5/2} - 3p^5h^{5/2}\}$$

Therefore

$$45\,|E| \leq h^{5/2}\,\|f''\|\{5(q-p)(5k^2+5k+1+pq(4k+2))$$

$$- 10((k+1)q-kp)(2k+1+pq) + 9(k+1)^2q - 9k^2p\}$$

Simplifying further, we have

$$45\,|E| \leq h^{5/2}\|f''\|\{q(5(5k^2+5k+1)-5k(4k+2)$$

$$-\,10(k+1)(2k+1) + 10k^2+9(k+1)^2)$$

$$+\,p(-5(5k^2+5k+1) + 5(k+1)(4k+2)$$

$$+\,10k(2k+1) - 10(k+1)^2 - 9k^2)\}$$

$$= h^{5/2}\|f''\|\{q(25k^2+25k+5-20k^2-10k$$

$$-\,20k^2-30k-10+10k^2+9k^2+18k+9)$$

$$+\,p(-25k^2-25k-5+20k^2+30k$$

$$+10+20k^2+10k-10k^2-20K-10-9k^2)\}$$

Therefore

$$45\,|E| \leq h^{5/2}\|f''\|\{Aq - Bp\}$$

where

$$A = 4k^2 + 3k + 4$$

and

$$B = 4k^2 + 5k + 5$$

But

$$Aq - Bp = \frac{(Aq-Bp)(Aq+Bp)}{Aq + Bp}$$

$$= \frac{A^2q^2 - B^2p^2}{Aq + Bp}$$

$$= \frac{(k+1)A^2 - kB^2}{Aq + Bp}$$

Now

$$A^2 = (4k^2+3k+4)^2 = 16k^4 + 24k^3 + 41k^2 + 24k + 16$$

and

$$B^2 = (4k^2+5k+5)^2 = 16k^4 + 40k^3 + 65k^2 + 50k + 25$$

Therefore

$$(k+1)A^2 = 16k^5 + 40k^4 + 65k^3 + 65k^2 + 40k + 16$$

and

$$kB^2 = 16k^5 + 40k^4 + 65k^3 + 50k^2 + 25k$$

We therefore have

$$(k+1)A^2 - kB^2 = 15k^2 + 15k + 16$$

We can now summarize the results of this section:

$$\int_a^{a+h} \frac{f(t)}{\sqrt{t}}\, dt = \frac{2\sqrt{h}f(x)}{p + q} + E \quad (a > 0)$$

where

$$k = a/h,\ p = \sqrt{k},\ q = \sqrt{k+1}$$

$$x = \frac{h}{3}(2k+1+pq)$$

and

$$|E| \leq {}^1/_{45}\, h^{5/2} \| f'' \| \left(\frac{C}{Aq+Bp}\right)$$

where

$$A = 4k^2 + 3k + 4$$

$$B = 4k^2 + 5k + 5$$

and

$$C = 15k^2 + 15k + 16$$

An approximate bound on the local <u>relative</u> error is given by

$$R \lesssim \frac{p + q}{2h^{1/2}\, |f(x)|} \cdot {}^1/_{45} h^{5/2} \| f'' \| \left(\frac{C}{Aq+Bp}\right)$$

$$= {}^1/_{90} h^2 \frac{\| f'' \|}{|f(x)|} \left(\frac{C(p+q)}{Aq+Bp}\right)$$

Letting

$$Q(k) = \frac{C(p+q)}{Aq + Bp}$$

we see that

$$Q(0) = 4 \text{ and } Q(\infty) = 3.75.$$

Hence, R is relatively independent of k and we have approximately

$$R \lesssim 1/23\, h^2 \frac{\|f''\|}{|f(x)|}$$

INTEGRAL INVERSION

In this section, we will compute the inverse of the integral of a positive, continuous, piecewise linear function. In subsequent sections, we will have two distinct applications for this process. We begin with the points

$$(x_i, f_i) \quad 1 \leq i \leq n$$

where $f_i > 0$. Define the continuous, piecewise linear function f on the i^{th} subinterval such that

$$f_i(x) = f_i + \frac{x - x_i}{h_i} (f_{i+1} - f_i) \quad x_i \leq x \leq x_{i+1}$$

where $h_i = x_{i+1} - x_i$. Therefore

$$f_i(x_i) = f_i$$

and

$$f_i(x_{i+1}) = f_{i+1}$$

The integral of this function is given by

$$F(x) = \int_{x_1}^{x} f(t)dt$$

Our goal is to compute the inverse of F (F^{-1}), i.e., if F(a) = b, then a = $F^{-1}(b)$. Now, if

$$x_i \leq x \leq x_{i+1}$$

$$F(x) = \int_{x_1}^{x_i} f(t)dt + \int_{x_i}^{x} f(t)dt$$

$$= F(x_i) + \int_{x_i}^{x} f_i + \frac{t - x_i}{h_i} (f_{i+1} - f_i)dt$$

$$= F_i + (x-x_i)f_i + \frac{(x-x_i)^2}{2h_i}(f_{i+1}-f_i)$$

Therefore, if we are given F such that $F_i \leq F \leq F_{i+1}$, we need only solve the following quadratic equation for x in order to invert the integral:

$$\frac{(f_{i+1}-f_i)}{2h_i}(x-x_i)^2 + f_i(x-x_i) - (F-F_i) = 0$$

Therefore

$$x - x_i = \frac{-f_i \pm (f_i^2+2/h_i(f_{i+1}-f_i)(F-F_i))^{\frac{1}{2}}}{\frac{f_{i+1} - f_i}{h_i}}$$

If $f_{i+1} > f_i$, we need the plus sign in order to get the positive root. If $f_{i+1} < f_i$, we still need the plus sign in order to get the smallest root. Therefore

$$x - x_i = \frac{(\sqrt{D}-f_i)h_i}{f_{i+1} - f_i}$$

where

$$D = f_i^2 + \frac{2}{h_i}(f_{i+1}-f_i)(F-F_i)$$

Noting that

$$F_{i+1} - F_i = \frac{h_i}{2}(f_i+f_{i+1})$$

D can be written

$$D = f_i^2 + \frac{2}{h_i}(f_{i+1}-f_i)(F-F_i) \cdot \frac{h_i/2(f_i+f_{i+1})}{F_{i+1} - F_i} = f_i^2 + (f_{i+1}^2-f_i^2)(\frac{F-F_i}{F_{i+1}-F_i})$$

Letting

$$\rho = \frac{F - F_i}{F_{i+1} - F_i}$$

clearly

$$0 \leq \rho \leq 1$$

and

$$D = f_i^2 + (f_{i+1}^2 - f_i^2)\rho = (1-\rho)f_i^2 + \rho f_{i+1}^2$$

therefore

$$D > 0$$

Now it can easily happen that $f_i = f_{i+1}$, therefore

$$x - x_i = \frac{(\sqrt{D}-f_i)h_i}{f_{i+1} - f_i}$$

needs further simplification.

$$x - x_i = \frac{(\sqrt{D}-f_i)h_i}{f_{i+1} - f_i} \cdot \frac{\sqrt{D} + f_i}{\sqrt{D} + f_i}$$

$$= \frac{(D-f_i^2)h_i}{(f_{i+1}-f_i)(\sqrt{D}+f_i)}$$

$$= \frac{((1-\rho)f_i^2+\rho f_{i+1}^2-f_i^2)h_i}{(f_{i+1}-f_i)(\sqrt{D}+f_i)}$$

$$= \frac{\rho h_i(f_{i+1}^2-f_i^2)}{(f_{i+1}-f_i)(\sqrt{D}+f_i)}$$

Therefore

$$x - x_i = \frac{\rho h_i(f_i+f_{i+1})}{\sqrt{D} + f_i}$$

$$= \frac{2\rho(F_{i+1}-F_i)}{f_i + \sqrt{D}}$$

$$= 2\rho(F-F_i) \cdot \frac{F_{i+1} - F_i}{F - F_i}/(f_i+\sqrt{D})$$

$$= \frac{2(F-F_i)}{f_i + \sqrt{D}}$$

We therefore have the result: if f is positive, continuous, and piecewise linear, and

$$F(x) = \int_{x_1}^{x} f(t)dt \qquad x_1 \leq x \leq x_n$$

and F^* is given such that

$$0 = F_1 \leq F^* \leq F_n$$

then

$$F^{-1}(F^*) = x^* = x_i + \frac{2(F^*-F_i)}{f_i + \sqrt{D}}$$

where

$$D = (1-\rho)f_i^2 + \rho f_{i+1}^2$$

$$\rho = \frac{F^* - F_i}{F_{i+1} - F_i}$$

$$F_i = \sum_{j=1}^{i-1} \frac{h_j}{2} (f_j+f_{j+1}) \qquad 2 \leq i \leq n$$

and i is defined by

$$F_i \leq F^* \leq F_{i+1}$$

Note that since $h_i > 0$ and $f_i > 0$, we always have $F_{i+1} > F_i$.

AUTOMATIC PIECEWISE LINEAR APPROXIMATION

In the process of generating the data which the filament winder understands, occasions arise when we must take a known (computable) function and generate points on the function which yield a good piecewise linear approximation to the function. The simplest way of doing this is to generate a set of equally-spaced values for the independent variable and evaluate the function on this uniform mesh. This is a poor method to use here, however, because the

functions we are dealing with can have regions of smooth, regular behavior followed by regions of abrupt, irregular behavior. A uniform mesh will evaluate the function too often in regions of low curvature and not often enough in regions of high curvature. The resulting piecewise linear approximations will therefore be excessively accurate in regions of low curvature and not accurate enough in regions of high curvature. The problem of generating good piecewise linear approximations becomes even more significant when there are microprocessor hardware or software limitations on how many data points you may use in your digitally controlled filament winder.

The technique used here to generate good piecewise linear functional approximations is the method of approximate error equidistribution discussed in general in Reference 1.

The idea is to create a mesh for the independent variable (x here) which approximately equidistributes the error in piecewise linear interpolation. The error bound for linear interpolation on the i^{th} subinterval is

$$\frac{h_i^2}{8} \|f''\|_{(i)}$$

where

$$h_i = x_{i+1} - x_i$$

and

$$\|f''\|_{(i)} = \max_{x_i \leq x \leq x_{i+1}} |f''(x)|$$

We would like to select the mesh such that

$$h_i^2 \|f''\|_{(i)} = \text{constant}$$

[1] C. deBoor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.

or such that

$$h_i \|f''\|_{(i)}^{\frac{1}{2}} = \text{constant}$$

This is equivalent to

$$\int_{x_i}^{x_{i+1}} \|f''\|_{(i)}^{\frac{1}{2}} dx = \text{constant}$$

Now, asymptotically as $h_i \to 0$,

$$\left| \|f''\|_{(i)}^{\frac{1}{2}} - | f''(x) |^{\frac{1}{2}} \right| \to 0$$

for any x in the i^{th} subinterval.

We therefore determine the mesh by selecting the x's for which

$$\int_{x_i}^{x_{i+1}} | f''(x) |^{\frac{1}{2}} dx = \text{constant}$$

Let

$$| f''(x) |^{\frac{1}{2}} = g(x)$$

and

$$\int_{x_i}^{x_{i+1}} g(x)dx = c$$

Therefore, if

$$G(x) = \int_{x_1}^{x} g(t)dt$$

then

$$G(x_i) = (i-1)c$$

and

$$G(x_n) = (n-1)c$$

We therefore have

$$\frac{G(x_i)}{G(x_n)} = \frac{i - 1}{n - 1}$$

or

$$x_i = G^{-1}\left(\frac{i-1}{n-1} G(x_n)\right)$$

where G^{-1} is the inverse of G.

A simple example is perhaps in order here. Suppose we wished to approximate x^p $(p > 0)$ by a piecewise linear function over the interval $(0, x_n)$.

$$f(x) = x^p$$

$$f'(x) \propto x^{p-1}$$

$$f''(x) \propto x^{p-2}$$

$$| f''(x) |^{\frac{1}{2}} = g(x) \propto x^{p/2-1}$$

$$G(x) \propto \int_0^x t^{p/2-1} dt \propto x^{p/2}$$

Therefore

$$\frac{G(x_i)}{G(x_n)} = \frac{i - 1}{n - 1} = \left(\frac{x_i}{x_n}\right)^{p/2}$$

and hence

$$x_i = \left(\frac{i-1}{n-1}\right)^{2/p} x_n$$

It is easy to see that if $p < 2$, the resulting mesh will tend to cluster more points near the left-hand side of the interval, while if $p > 2$, more points will cluster on the right. If $p = 2$, the mesh is uniform because $f''(x)$ is constant.

Usually, we do not know $f''(x)$, because f may only be computable and not given as a simple explicit function of x. We can, however, easily estimate $f''(x)$ over a uniform mesh using finite differences. We may, therefore, estimate g on a uniform mesh and define our g (estimate) to be piecewise linear. G will then be piecewise quadratic and invertible by the technique discussed in the previous section on integral inversion.

If we let x_i be the i^{th} old, uniform mesh point and x_k* be the k^{th} new, error equidistributing mesh point, we may use the results on integral inversion to define the new mesh:

$$x_1^* = x_1$$

$$x_n^* = x_n$$

$$x_k^* = x_i + \frac{2(G_k^*-G_i)}{g_i + \sqrt{D}} \qquad 2 \leqslant k \leqslant n-1$$

where

$$G_k^* = \frac{k - 1}{n - 1} G_n$$

$$\rho = \frac{G_k^* - G_i}{G_{i+1} - G_i}$$

$$D = (1-\rho)g_i^2 + \rho g_{i+1}^2$$

and i is defined by

$$G_i \leqslant G_k^* \leqslant G_{i+1}$$

The g's are defined by

$$g_i = | f_i'' |^{1/2}$$

where f_i'' is defined by

$$f_i'' = \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2}$$

and h is the uniform mesh spacing.

The only minor problem that can arise in this process is when some consecutive values of g are zero (and some consecutive values of G are equal) in a perfectly linear section of f. When there is at least one nonzero g_i, however, the problem of $G_i = G_k^* = G_{i+1}$ occurs with probability zero. We can therefore avoid this problem by simply setting $x_k^* = x_i$ if $G_k^* = G_i$ and not compute ρ.

If we have some idea of how accurate we would like our piecewise linear approximation over the new mesh to be, we may estimate the needed number of points in the new mesh in the following manner. Suppose we desire an absolute error tolerance E. We would like

$$E = \frac{1}{8} h^2 g^2 = \text{constant}$$

where

$$g(x) = |f''(x)|^{1/2}$$

as before.

In particular, we would like

$$E = \frac{1}{8} h_{min}^2 g_{max}^2$$

Now there will be an i in the new mesh for which $x_{i+1}^* - x_i^* = h_{min}$ will hold.

But we already know that

$$G(x_{i+1}^*) - G(x_i^*) = G(x_n)/(n-1)$$

and that

$$G(x_{i+1}^*) - G(x_i^*) = \int_{x_i^*}^{x_{i+1}^*} g(x)dx \sim h_{min} g_{max}$$

Hence, we have approximately that

$$h_{min} = \frac{G(x_n)}{(n-1)g_{max}}$$

and

$$E = \frac{1}{8} \frac{G(x_n)^2}{(n-1)^2 g_{max}^2} \cdot g_{max}^2$$

Solving for n gives us

$$n \approx 1 + \frac{G(x_n)}{\sqrt{8E}}$$

PATH COMPUTATION

In this section, we compute the actual geodesic (or quasi-geodesic) path on the surface of revolution. That is, we obtain the relation between x and θ. Given the radius or profile function $r(x)$ and the polar radius function $r_o(x)$, we obtain θ by integrating:

$$\theta'(x) = \frac{r_0(x)}{r(x)} \left(\frac{1+r'(x)^2}{r(x)^2-r_0(x)^2}\right)^{\frac{1}{2}}$$

We have already shown that $\theta'(x)$ has square root singularities at the left and right turning points, and we have derived a quadrature formula with error term to do the integrations and check accuracy. We apply the quadrature formula in the following manner:

For x nearer to the left turning point (x=0),

$$\theta(x+h) - \theta(x) = \int_x^{x+h} \theta'(t)dt = \int_x^{x+h} \frac{\sqrt{t}\theta'(t)}{\sqrt{t}} dt = \int_a^{a+h} \frac{f(t)}{\sqrt{t}} dt$$

where $a = x$ and $f(t) = \sqrt{t}\theta'(t)$.

For x nearer to the right turning point (x=L),

$$\theta(x+h) - \theta(x) = \int_x^{x+h} \theta'(t)dt = \int_x^{x+h} \frac{\sqrt{L-t}\theta'(t)}{\sqrt{L-t}} dt$$

$$= \int_{x-L}^{x+h-L} \frac{\sqrt{-t}\theta'(t+L)}{\sqrt{-t}} dt$$

$$= -\int_{L-x}^{L-x-h} \frac{\sqrt{t}\theta'(L-t)}{\sqrt{t}} dt$$

$$= \int_{L-x-h}^{L-x} \frac{\sqrt{t}\theta'(L-t)}{\sqrt{t}} dt$$

$$= \int_a^{a+h} \frac{f(t)}{\sqrt{t}} dt$$

where $a = L-x-h$ and $f(t) = \sqrt{t}\theta'(L-t)$.

Up to this point, we have been using x as the independent variable in this analysis. This is the obvious choice, since the radius function r and the polar radius function r_0 must be described as functions of x. At this point, however, we will switch to using θ as our independent variable because x as a function of

θ has no singularities, can therefore be subjected to higher order interpolation, and is a single-valued function for an entire circuit (left turning point to right and back to left again). In order to define x as a function of θ accurately, we proceed with the following steps:

1. Generate a first x mesh which is uniform.
2. Integrate θ' over this x mesh to give the first θ mesh.
3. Use this θ,x data to generate a second θ mesh which equidistributes the error in piecewise linear interpolation of x.
4. Use piecewise cubic interpolation of the first set of θ,x data and the second θ mesh to generate a second x mesh.
5. Integr te θ' again with respect to the second x mesh and produce the third and final θ mesh.
6. Define x as a function of θ using piecewise cubic interpolation of the second x mesh with respect to the third θ mesh.

In what follows, let dot ($\cdot$) denote $d/d\theta$ and let n be the number of points in each mesh. Ordinarily, higher order interpolation would require a higher order derivative than the second for the process of error equidistribution, but if we define the nodal derivatives for the interpolating piecewise cubics in the following manner:

$$\dot{x}(\theta_1=0) = \dot{x}(\theta_n) = 0$$

$$\dot{x}(\theta_i) = \frac{x_{i+1} - x_{i-1}}{\theta_{i+1} - \theta_{i-1}} \qquad (1 < i < n)$$

then the cubics will be only $O(h^2)$ accurate (same as linear) for a nonuniform mesh, and hence, a good mesh for piecewise linear approximation will also be a good mesh for this particular piecewise cubic approximation. Another reason for

defining the nodal derivatives this way is to produce a more stable (in the sense of preserving monotonicity) cubic interpolant over nonuniform meshes. The use of a piecewise cubic to define $x(\theta)$ turns out to be numerically critical around the turning points; if we do not have $\dot{x}(o) = \dot{x}(\theta_n) = 0$, we get a noticeable kink in our winding data. The piecewise cubic interpolant is given by

$$x(\theta) = x_i(1-3\rho^2+2\rho^3) + x_{i+1}(3\rho^2-2\rho^3)$$
$$+ (\theta_{i+1}-\theta_i)(\dot{x}_i(\rho-2\rho^2+\rho^3) + \dot{x}_{i+1}(-\rho^2+\rho^3))$$

where

$$\theta_i \leqslant \theta \leqslant \theta_{i+1}$$

$$\rho = \frac{\theta - \theta_i}{\theta_{i+1} - \theta_i}$$

$$\dot{x}_1 = \dot{x}_n = 0$$

and

$$\dot{x}_i = \frac{x_{i+1} - x_{i-1}}{\theta_{i+1} - \theta_{i-1}} \qquad (1 < i < n)$$

PATH WINDER RELATIONS

The (2-axis) filament winding machine or winder consists basically of the surface, which is rotated on its axis by the winder and a carriage which pays out a taut filament from delivery point D, which moves parallel to and at a sufficient distance from the axis of the surface. The task at hand is to coordinate the rotational movement of the surface and translational carriage movement in such a way as to lay down the filament on a predetermined path on the surface. This is accomplished ultimately by computing carriage position as a function of surface rotation. This is the only data that the winder "understands." For purposes of visualization, it is best to think of the

surface and the desired path on it as being stationary and the carriage as both translating parallel to the surface axis and rotating about the surface.

Given a point P on the surface path, the tangent plane is formed by the vector tangent to the path and the vector tangent to the meridian.

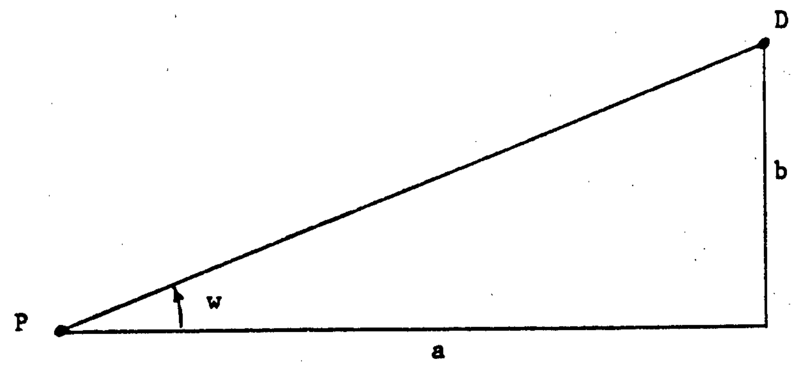

Figure 8. Tangent plane relations.

The taut filament emanating from the delivery point D just meets the path at point P. The winding angle w is given by

$$\sin w = \frac{r_o}{r} \quad \text{or} \quad \tan w = \frac{r_o}{\sqrt{r^2 - r_o^2}}$$

We therefore have

$$\frac{b}{a} = \frac{r_o}{\sqrt{r^2 - r_o^2}} \quad \text{or} \quad b^2 = \frac{a^2 r_o^2}{r^2 - r_o^2}$$

The meridian plane is formed by the tangent to the meridian and the axis of the surface. ℓ is the x distance by which the delivery point D leads (on the first half circuit) the path point P. In this picture, r' is assumed to be positive.

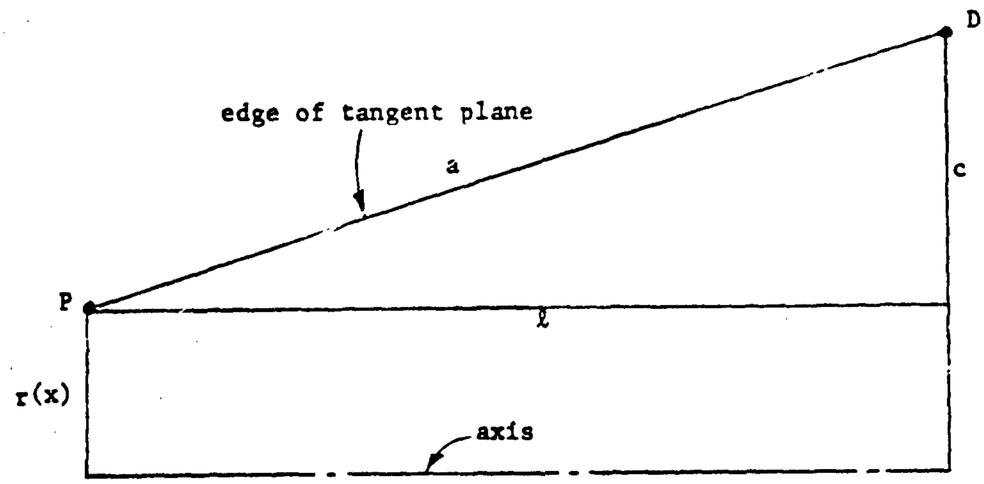

Figure 9. Meridian plane relations.

We immediately have the relations

$$\frac{c}{\ell} = r' \quad \text{and} \quad a^2 = \ell^2 + c^2$$

Therefore

$$a^2 = \ell^2 + r'^2\ell^2$$
$$= \ell^2(1+r'^2)$$

and

$$b^2 = \frac{a^2 r_o^2}{r^2 - r_o^2} = \frac{r_o^2 \ell^2(1+r'^2)}{r^2 - r_o^2} = (r\ell\theta')^2$$

The hoop plane is formed by the hoop or parallel through P. The lead angle λ is given by

$$\cos\lambda = \frac{r + c}{f}$$

where f is the filament delivery height or distance from the surface axis to the delivery point D.

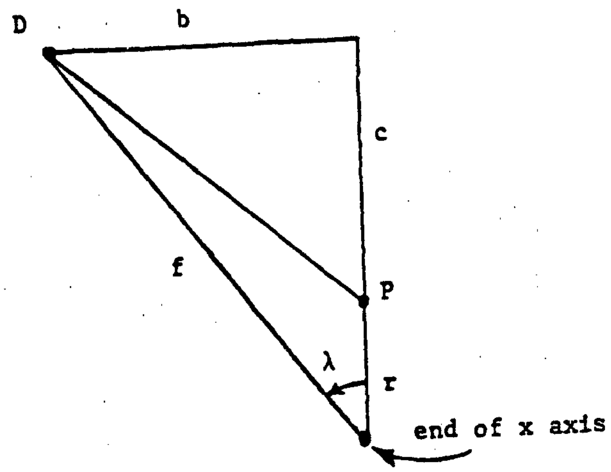

Figure 10. Hoop plane relations.

Since $c = \ell r'$, we have

$$\cos \lambda = \frac{r + \ell r'}{f}$$

We also have the relation

$$f^2 = b^2 + (c+r)^2$$

$$= (r\ell\theta')^2 + (\ell r'+r)^2$$

$$= (r^2\theta'^2+r'^2)\ell^2 + 2rr'\ell + r^2$$

The lead distance ℓ is therefore given implicitly by the following quadratic equation:

$$(r^2\theta'^2+r'^2)\ell^2 + 2rr'\ell + r^2 - f^2 = 0$$

Solving for ℓ, we have

$$\ell = \frac{-2rr' \pm \sqrt{4r^2r'^2-4(r^2-f^2)(r^2\theta'^2+r'^2)}}{2(r^2\theta'^2+r'^2)}$$

$$= \frac{-rr' \pm \sqrt{(fr')^2 + (f^2-r^2)(r\theta')^2}}{(r\theta')^2 + r'^2}$$

Note that if $r' = 0$, ℓ and θ' have the same sign. Therefore,

$$\ell = \frac{-rr' + \frac{\theta'}{|\theta'|}\sqrt{(fr')^2 + (f^2-r^2)(r\theta')^2}}{(r\theta')^2 + r'^2}$$

$$= \frac{-rr' + \theta'\sqrt{r^2(f^2-r^2) + (fr'/\theta')^2}}{(r\theta')^2 + r'^2}$$

The lead distance is therefore given by

$$\ell = \frac{(D^{1/2}-rr'/\theta')/\theta'}{r^2+(r'/\theta')^2}$$

where

$$D = r^2(f^2-r^2) + (fr'/\theta')^2$$

This formula for ℓ is most accurate when $r'/\theta' < 0$. If $r'/\theta' > 0$, we rationalize:

$$\ell = \frac{(D^{1/2}-rr'/\theta')/\theta'}{r^2 + (r'/\theta')^2} \cdot \frac{D^{1/2} + rr'/\theta'}{D^{1/2} + rr'/\theta'}$$

$$= \frac{(D-(rr'/\theta')^2)/\theta'}{(r^2+(r'/\theta')^2)(D^{1/2}+rr'/\theta)}$$

but

$$D - (rr'/\theta')^2 = r^2(f^2-r^2) + (fr'/\theta')^2 - (rr'/\theta')^2$$

$$= (f^2-r^2)(r^2+(r'/\theta')^2)$$

Therefore

$$\ell = \frac{(f^2-r^2)/\theta'}{rr'/\theta' + D^{1/2}} \qquad (r'/\theta' > 0)$$

We have used the reciprocal of θ' in these formulas because it is always finite. Having established ℓ in terms of the filament delivery height and the path, we may compute λ from

$$\cos\lambda = \frac{r + \ell r'}{f}$$

Finally, the carriage rotation R (relative to the delivery point) is given by

$$R = \theta + \lambda$$

and the traversing carriage position is given by

$$T = x + \ell$$

Subsequently, it will be necessary to express T as an explicit function of R - the only data the winder "understands" directly.

WINDER DATA GENERATION

The process of generating the data that the winder understands (carriage traverse position T as a function of carriage rotation R) is complicated by the fact that T is not directly expressible or computable as a function of R. Instead, R and T are both parametric functions of θ, since given θ, x can be computed and given x, ℓ and λ may be computed. The solution to this problem is to find a θ mesh which will yield an R mesh which will ultimately equidistribute the error in T as a function of R. We will let dot ($\cdot$) denote differentiation with respect to θ in what follows. Since our goal is to equidistribute the error in T as a function of R, we first write down the approximate condition embodying this requirement

$$(dR)^2 \left| \frac{d^2T}{dR^2} \right| = \text{constant} \approx 8E$$

but

$$dR = \dot{R}d\theta$$

We therefore want

$$(\dot{R}d\theta)^2 \left| \frac{d^2T}{dR^2} \right| = \text{constant}$$

but

$$\frac{dT}{dR} = \frac{dT}{d\theta}\frac{d\theta}{dR} = \dot{T}/\dot{R}$$

and

$$\frac{d^2T}{dR^2} = \frac{d}{dR}\,(\dot{T}/\dot{R}) = \frac{d}{d\theta}\,(\dot{T}/\dot{R})\,\frac{d\theta}{dR} = (\ddot{T}\dot{R}-\dot{T}\ddot{R})/\dot{R}^3$$

Hence

$$(\dot{R}d\theta)^2\,\left|(\ddot{T}\dot{R}-\dot{T}\ddot{R})/\dot{R}^3\right| = (d\theta)^2\,\left|\ddot{T}-\dot{T}\ddot{R}/\dot{R}\right| = \text{constant}$$

Therefore, our g function for the θ mesh is

$$g(\theta) = \left|\ddot{T} - \dot{T}\ddot{R}/\dot{R}\right|^{1/2}$$

Using this g function in our usual error equidistributing process will yield a θ mesh which will produce an R mesh which will equidistribute the error in T as a function of R. The major difficulty first seems to be computing the derivatives in g, and although this would indeed be a cumbersome task to do analytically, it is a simple task to do numerically. For instance, using central differences, we can obtain the following after some simplification:

$$\ddot{T}_i - \dot{T}_i\ddot{R}_i/\dot{R}_i = (2/h^2)\{(T_{i+1}-T_i)(R_i-R_{i-1}) - (T_i-T_{i-1})(R_{i+1}-R_i)\}/(R_{i+1}-R_{i-1})$$

We proceed algorithmically as follows:

1. Compute R and T over a dense, uniform θ mesh. For each θ value, compute x, ℓ, and λ and store $\theta+\lambda$ and $x+\ell$ in R and T, respectively.
2. Using finite differences, compute a piecewise linear approximation to g.
3. Invert the integral of g to obtain a new θ mesh.
4. Evaluate R and T again over this new θ mesh.

To be usable, the R/T data generated by this process must represent a single-valued function. This will be the case if the polar radius function has been properly defined to lie between the radius function and the polar radius lower bound function. If the lower bound function exceeds the polar radius function anywhere however, it is likely that the R data will not be monotonically increasing, causing unacceptable loops or cusps in the R/T curve.

TIME BASE COMPUTATION

Geometrically speaking, the function $T(R)$ is all that is needed to wrap the surface; but practically speaking, we must also decide how R is to behave as a function of time (or vice versa). Let us compute the velocity of the delivery point relative to the surface. The components of the position vector of the delivery point are

$$x = T(R)$$

$$y = f \sin R$$

$$z = f \cos R$$

where f is the filament delivery height.

The components of velocity are therefore

$$\dot{x} = T'(R)\dot{R}$$

$$\dot{y} = f \cos R\dot{R}$$

$$\dot{z} = -f \sin R\dot{R}$$

where $(\cdot)$ denotes $\frac{d}{dt}$. The square of the speed is

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$= T'(R)^2\dot{R}^2 + f^2\dot{R}^2 \cos^2 R + f^2\dot{R}^2 \sin^2 R$$

$$= (T'(R)^2+f^2)\dot{R}^2$$

and the speed is

$$v = (T'(R)^2+f^2)^{1/2}\dot{R}$$

If R is a linear function of t, $\dot{R}$ is constant and $|\dot{x}|$ will follow $|T'(R)|$. In order to make $|\dot{x}|$ sufficiently small everywhere, we may need to make $\dot{R}$ so unacceptably small that we will slow down the winding process far too much when $|T'(R)|$ is small. Clearly, we need $\dot{R}$ small when $|T'(R)|$ is large and vice versa. We obtain a variable $\dot{R}$ by rewriting the previous equation as

$$dt = \frac{1}{v} (T'(R)^2+f^2)^{1/2}dR$$

Integrating, we get

$$t_{i+1} - t_i = \int_{R_i}^{R_{i+1}} \frac{1}{v} (T'(R)^2+f^2)^{½}dR$$

Approximating this integral by the midpoint rule and using differences gives us the following time base recursion formula, where the positive speed function v is anything we wish:

$$t_{i+1} - t_i \approx \frac{2}{v_i + v_{i+1}} ((T_{i+1}-T_i)^2 + f^2(R_{i+1}-R_i)^2)^{½}$$

SURFACE COVERAGE RELATIONS

The mathematical machinery we have developed thus far enables us to define a good quasi-geodesic path on a surface of revolution and to successfully wind filament along this path by computing carriage traverse position as a single-valued function of surface rotation. It is very unlikely, however, that actually winding filament along such a path will result in a uniformly covered surface. Our task in this section is therefore to modify our nominal quasi-geodesic path slightly by multiplying θ' by some factor α, near unity, which will guarantee that repeated application of the winding data subsequently produced from the modified path will serve to cover the surface uniformly.

We begin by noting the effect on the polar radius function of multiplying θ' by α. The effective polar radius function will be denoted by ρ_o. Using the general definition of θ, we can write the following equivalence:

$$\frac{\alpha r_o}{r} \left(\frac{1+r'^2}{r^2-r_o^2}\right)^{½} = \frac{\rho_o}{r} \left(\frac{1+r'^2}{r^2-\rho_o^2}\right)^{½}$$

Solving this equation for ρ_o gives us:

$$\rho_o = \frac{\alpha r r_o}{(r^2+(\alpha^2-1)r_o^2)^{½}}$$

Note that if $\alpha = 1$, $\rho_o = r_o$, and if $\alpha = \infty$, $\rho_o = r$. It is easy to prove that if $\alpha \geqslant 1$, then $r_o \leqslant \rho_o \leqslant r$. If $\alpha < 1$, we may risk violating the polar radius lower bound function. Remember that this lower bound function is established once and for all by the radius function r.

We now define a couple of discontinuous functions for reference.

$$FL(x) = \text{Floor of } x = \text{the greatest integer} \leqslant x$$

$$CE(x) = \text{Ceiling of } x = \text{the least integer} \geqslant x = -FL(-x)$$

Note that if fl(x) is the same as FL(x), but defined only for positive x, then the following defines FL(x) for all x:

$$FL(x) = fl(x+n) - n$$

where

$$n = fl(|x|+1)$$

We must now investigate the closure properties of our paths. For instance, if one circuit of our nominal path wraps through a rational multiple of 2π radians, the path is guaranteed to ultimately close (i.e., return to exactly the same point on the polar parallel it started at). If the total wrap $(2\theta(L))$ of a nominal circuit is given by

$$w_{NC} = 2\pi\ (i + \frac{j}{k})$$

where j/k is in lowest terms, it becomes obvious that the path will repeat itself for the first time after exactly k circuits. If j/k were not in lowest terms, j and k would have a greatest common factor (GCF) greater than unity. Repetition would therefore begin after only k/GCF(j,k) circuits. For the nominal path, k could naturally be quite large. We will establish uniformly wrapping paths with relatively small values of k. In what follows, we will give special names and designations to i, j, and k.

We will call i the number of complete revolutions per circuit and denote it by N_{RC}. We will call j the return index, separation index, or number of separations per circuit and will denote it by N_{SC}. We will call k the circuit index or number of circuits per repetition and will denote it by N_{CR}. In what follows, we will also allow a somewhat looser interpretation of the term "repetition." For a uniformly wrapping path, we will say that the first repetition has occurred when the band of filaments begins winding just alongside and slightly overlapping the first circuit. Hence, a repetition may or may not be closed. We will always use the term "closed" to mean exact return to the initial point.

The wrap of a circuit in a closed repetition is given by

$$w_{CCR} = 2\pi \left(N_{RC} + \frac{N_{SC}}{N_{CR}}\right)$$

The wrap for the entire closed repetition is, of course,

$$w_{CR} = N_{CR} \cdot w_{CCR} = 2\pi(N_{CR} \cdot N_{RC} + N_{SC})$$

If we actually wind a filament along this path, the windings will naturally close after N_{CR} circuits. The pattern of filament bands on the surface will be such that neighboring left to right travelling bands will be separated from each other by a wrap angle of exactly $2\pi/N_{CR}$. Neighboring points of tangency of the bands to the polar parallels will also be separated by the same angle. We therefore refer to $2\pi/N_{CR}$ as the separation angle. The angle $2\pi\ N_{SC}/N_{CR}$ will be called the return angle - the angle between the band's point of tangency to the left polar hoop at the beginning of a circuit and the band's point of tangency to the left polar hoop at the end of a circuit. The return angle is obviously the product of the separation angle and the separation index. We naturally

always select values of N_{CR} and N_{SC} which are relatively prime, i.e., values for which GCF $(N_{SC}, N_{CR}) = 1$. Another quantity which we must introduce is the number of circuits per separation, denoted by N_{CS}. This quantity is the number of circuits which must be wound in order to obtain a net wrap advance of exactly one separation angle beyond the starting point. Consider the following diagram, where A is the starting point on the polar hoop, B is the return point on the polar hoop (reached after one circuit), and C is a point on the polar hoop which is advanced exactly one separation angle beyond point A.

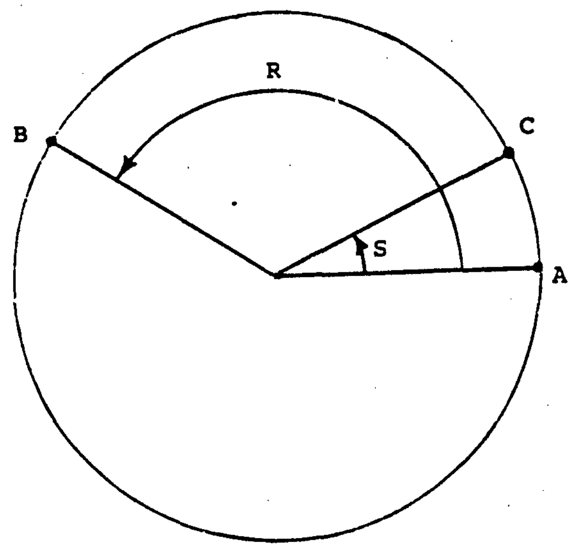

Figure 11. Return and separation angles for a closed repetition.

S is the separation angle and R is the return angle. Now, after N_{CS} circuits, the path will be tangent to the polar hoop at C. We therefore have

$$N_{CS} \cdot R = 2\pi k + S$$

but

$$S = \frac{2\pi}{N_{CR}} \quad \text{and} \quad R = N_{SC} \cdot \frac{2\pi}{N_{CR}}$$

Therefore

$$N_{CS} \cdot N_{SC} \cdot 2\pi / N_{CR} = 2\pi k + 2\pi / N_{CR}$$

and

$$N_{CS} \cdot N_{SC} = 1 + k \cdot N_{CR}$$

or

$$N_{CS} \cdot N_{SC} \equiv 1 \bmod N_{CR}$$

Hence, N_{CS} and N_{SC} are "reciprocals" in the modulo (or congruence) arithmetic restricted to integers zero through $N_{CR}-1$, where multiples of N_{CR} are equivalent (congruent) to zero. We compute N_{CS} as the first positive integer for which

$$(N_{CS} \cdot N_{SC} - 1)/N_{CR}$$

has a zero remainder. This value of N_{CS} is guaranteed to exist, provided $GCF(N_{SC}, N_{CR}) = 1$ (ref 2).

We will now show what the number N_{CS} is used for. Assuming the value of N_{CR} is relatively small, a single closed repetition will not have covered much of the surface and we obviously won't cover any more of the surface by continuing to wind along a closed path. We now imagine the band of filaments to be cut at some point and one cut end to be displaced from the other through a small wrap angle which we will call the band advance per repetition or B_R. To get uniform coverage of the surface, we would like B_R to be such that an extension of the displaced cut end will continuously overlap the previously wound band ever so slightly. Suppose we were to wind a couple more repetitions, using the same B_R. What would the configuration of bands on the surface look like? There would still be N_{CR} separated right travelling bands on the surface, but they would be nearly three times as wide as the first band. Also, there would still be two cut ends, separated by a wrap angle of $3B_R$. Now, it should be clear that as the angular separation between the two cut ends approaches the separation angle, the surface will be almost completely covered by a layer of filament. We

[2] W. E. Deskins, Abstract Algebra, MacMillan Company, New York, 1964.

will now address the problem of obtaining exact closure of the path after the surface has been covered by one layer of filament. Consider the following diagram:

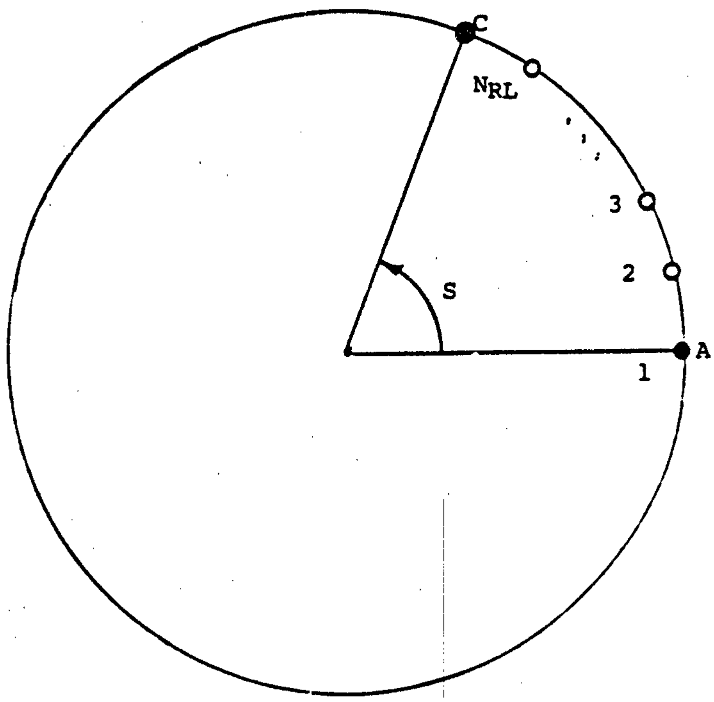

Figure 12. Starting points for unclosed repetitions.

The points labeled one through N_{RL} (number of repetitions per layer) are the starting points of the first through the N_{RL}th repetitions, respectively. Each point is separated from its neighbors by the band advance per repetition, B_R. Now, for the sake of uniformity, we would like to select B_R so that the angle between point C and the point indexed by N_{RL} is exactly equal to B_R; i.e., we would like our path to already be winding directly on top of itself when it reaches point C for the second time. Here, for unclosed repetitions, the separation angle S is slightly greater than it is for a closed repetition. Here

$$S = \frac{2\pi}{N_{CR}} + N_{CS} \cdot B_C$$

where B_C is the band advance per circuit. We therefore want

$$\frac{2\pi}{N_{CR}} + N_{CS} \cdot B_C = N_{RL} \cdot B_R$$

but

$$B_R = N_{CR} \cdot B_C$$

Therefore

$$\frac{2\pi}{N_{CR}} + N_{CS} \cdot B_C = N_{RL} \cdot N_{CR} \cdot B_C$$

Hence

$$B_C = \frac{2\pi}{N_{CR}(N_{CR} \cdot N_R - N_{CS})}$$

and

$$B_R = \frac{2\pi}{N_{CR} \cdot N_{RL} - N_{CS}}$$

We have thus computed the band advances necessary for our path to reach point C for the <u>second</u> time and <u>already</u> be winding directly on top of itself. Where and when did our path close? It closed at point A, N_{CS} circuits, before it reached point C. But it reached point C (for the second time) after $N_{CR} \cdot N_{RL}$ circuits. We see then that the exact, integral number of circuits per (closed) layer is given by

$$N_{CL} = N_{CR} \cdot N_{RL} - N_{CS}$$

$$= N_{CR}\left(N_{RL} - \frac{N_{CS}}{N_{CR}}\right)$$

$$= N_{CR}\left(N_{RL} - 1 + \frac{N_{CR} - N_{CS}}{N_{CR}}\right)$$

The exact, nonintegral number of repetitions per closed layer is therefore given by

$$N'_{RL} = N_{RL} - 1 + \frac{N_{CR} - N_{CS}}{N_{CR}}$$

where, of course, N_{RL} is the integral number of repetitions needed to close the layer and add N_{CS} additional circuits. Therefore, given N_{CR}, N_{SC}, and N_{RL}, we may compute N_{CS}, N_{CL}, and B_C.

We now show how to compute good values for N_{CR} and N_{SC}. Denoting the nominal return angle by R and the separation angle by S

$$2\theta(L) = 2\pi \cdot N_{RC} + R$$

Thus

$$\frac{\theta(L)}{\pi} = N_{RC} + \frac{R}{2\pi}$$

and

$$FL\left(\frac{\theta(L)}{\pi}\right) = N_{RC}$$

Hence, R is given by

$$R = 2(\theta(L) - \pi \cdot N_{RC}) = 2(\theta(L) - \pi FL(\theta(L)/\pi))$$

If we want $\alpha \geqslant 1$ (to avoid the polar radius lower bound function), we will need

$$R \leqslant S \cdot N_{SC}$$

where

$$S = 2\pi / N_{CR}$$

Hence

$$N_{SC} \geqslant R/S = N_{CR} \cdot R/2\pi$$

and we define N_{SC} by

$$N_{SC} = CE(N_{CR} \cdot R/2\pi)$$

(provided GCF $(N_{CR}, N_{SC}) = 1$). We also want $S \cdot N_{SC} - R$ to be as small as possible (in order that only minimal perturbation of our nominal quasi-geodesic will be required). We therefore use the following procedure to select nearly optimal values of N_{CR} and N_{SC}. For values of N_{CR} between some minimum and maximum value, compute:

$$N_{SC} = CE(N_{CR} \cdot R/2\pi)$$

If $GCF(N_{SC}, N_{CR}) = 1$, compute

$$\delta = N_{SC} \cdot 2\pi/N_{CR} - R$$

If this is the smallest value of δ yet encountered, save δ, N_{CR}, and N_{SC}. Continue on to the next value of N_{CR}.

The values of N_{CR} and N_{SC} obtained by this procedure will insure that the return angle for a circuit of a closed repetition will be as close as possible to the nominal return angle while still maintaining the condition $\alpha \geqslant 1$.

We now consider the specification of α. Our effective wrap angle function (τ) derivative is given by

$$\tau'(x) = \alpha\theta'(x)$$

$$\tau(x) = \int_0^x \tau'(t)dt = \int_0^x \alpha\theta'(t)dt$$

If α is a constant, we have

$$\tau(x) = \alpha\theta(x)$$

and

$$\tau(L) = \alpha\theta(L)$$

but we want

$$2\tau(L) = 2\pi(N_{RC} + \frac{N_{SC}}{N_{CR}}) + \beta_C$$

$$= 2\pi(N_{RC} + \frac{N_{SC}}{N_{CR}}) + \frac{2\pi}{N_{CR} \cdot N_{CL}}$$

Therefore

$$2\alpha\theta(L) = 2\pi(N_{RC} + \frac{N_{SC}}{N_{CR}}) + \frac{2\pi}{N_{CR} \cdot N_{CL}}$$

and

$$\alpha = \frac{\pi}{\theta(L)} (N_{RC} + \frac{N_{SC}}{N_{CR}} + \frac{1}{N_{CR} \cdot N_{CL}})$$

Up to this point, one may have been thinking of α as a constant, but it turns out that if α is a constant and r' is large somewhere, then ρ_o' will tend to follow r' and be large also. The path will thus deviate considerably from the nominal quasi-geodesic path. We may find the extent of this deviation intolerable, so we define α as a function:

$$\alpha(x) = 1 + \beta p(x)$$

where β is a constant. We call $p(x)$ the perturbation function and we make $p(x)$ zero when we want ρ_o to be exactly equal to r_o. We can still make p constant if we wish, but we now have more flexibility at a small computational price. For the new definition of α,

$$\tau'(x) = \alpha(x)\theta'(x) = (1+\beta p(x))\theta'(x)$$

$$\tau(x) = \int_o^x \tau'(t)dt = \int_o^x (1+\beta p(t))\theta'(t)dt$$

$$= \theta(x) + \beta\int_o^x p(t)\theta'(t)dt$$

Letting

$$P(x) = \int_o^x p(t)\theta'(t)dt$$

(which can be computed with our square root singularity quadrature formula), we have

$$\tau(x) = \theta(x) + \beta P(x)$$

and

$$\tau(L) = \theta(L) + \beta P(L)$$

so we can calculate β easily as soon as we know $\tau(L)$. Also,

$$\tau_i = \tau_{i-1} + \theta_i - \theta_{i-1} + \beta(P_i - P_{i-1})$$

defines the path data for our perfectly closed path in terms of our nominal path data, the perturbation function p and β.

Even though α is now a function, we still have the relation:

$$\tau(L) = \pi(N_{RC} + \frac{N_{SC}}{N_{CR}} + \frac{1}{N_{CR} \cdot N_{CL}})$$

$$= \pi(N_{RC} + \frac{N_{SC}}{N_{CR}} + \frac{1}{N_{CR}(N_{CR} \cdot N_{RL} - N_{CS})})$$

Since N_{RC}, N_{CR}, N_{SC}, and N_{CS} are computed on the basis of the nominal path, we need only specify N_{RL} in order to compute $\tau(L)$ and β. We establish N_{RL} from the requirement that a layer of filament must completely cover the surface for any x. We assume that our band of filaments has bandwidth b.

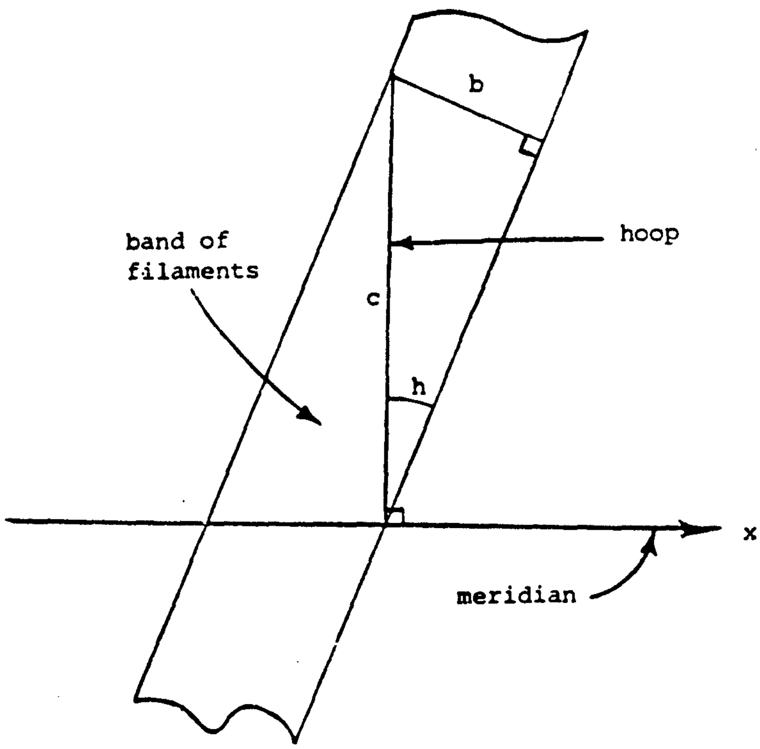

Figure 13. Effect of finite bandwidth.

From the figure, we have the approximation

$$b = c \sin h$$

(exact for r and r_o constant; not valid at all at turning points) and

$$b^2 = c^2 \sin^2 h = c^2(1-\cos^2 h)$$

but

$$\cos h = \frac{\rho_o(x)}{r(x)}$$

Hence

$$b^2 = c^2\left(1 - \left(\frac{\rho_o}{r}\right)^2\right)$$

but

$$\rho_o = \frac{\alpha r r_o}{(r^2+(\alpha^2-1)r_o^2)^{1/2}}$$

Hence

$$1 - \left(\frac{\rho_o}{r}\right)^2 = \frac{r^2-r_o^2}{r^2 + (\alpha^2-1)r_o^2}$$

We therefore have

$$b^2 = \frac{c^2(r^2-r_o^2)}{r^2+(\alpha^2-1)r_o^2}$$

Now, in order for one layer to completely cover the surface at this point, we clearly must have

$$c \cdot N_{CL} \geq 2\pi r$$

from which we conclude that we need

$$b^2 \geq \left(\frac{2\pi r}{N_{CL}}\right)^2\left(\frac{r^2-r_o^2}{r^2+(\alpha^2-1)r_o^2}\right)$$

but a sufficient condition for the satisfaction of the previous inequality is the following one:

$$b^2 \geq \left(\frac{2\pi r}{N_{CL}}\right)^2\left(\frac{r^2-r_o^2}{r^2}\right)$$

$$= \left(\frac{2\pi}{N_{CL}}\right)^2(r^2-r_o^2)$$

Hence

$$N_{CL}^2 \geq \left(\frac{2\pi}{b}\right)^2 (r^2 - r_o^2)$$

but we want this inequality to hold for all x. Therefore, if we define

$$M = \max_{0 \leq x \leq L} \left(r(x)^2 - r_o(x)^2\right)$$

then we need

$$N_{CL} \geq \frac{2\pi\sqrt{M}}{b}$$

and

$$N_{CR} \cdot N_{RL} - N_{CS} \geq \frac{2\pi\sqrt{M}}{b}$$

We therefore conclude that a value of N_{RL} sufficient to guarantee complete coverage of the surface everywhere is given by

$$N_{RL} = CE\left(\left(\frac{2\pi\sqrt{M}}{b} + N_{CS}\right)/N_{CR}\right)$$

From this value of N_{RL}, we compute $\tau(L)$ and β. If β is not particularly small, however, we may be able to get away with an even smaller value of N_{RL}. If a smaller value of N_{RL} gives complete coverage, we must have

$$m \leq 0$$

where

$$m = \max_{0 \leq x \leq L} \left(\frac{2\pi r}{N_{CL}}\right)^2 \left(\frac{r^2 - r_o^2}{r^2 + (\alpha^2 - 1) r_o^2}\right) - b^2$$

The global maxima M and m are easily computed with a couple of applications (global and local) of adaptive (error equidistributing) sampling.

SURFACE BUILDUP RELATIONS

In the previous section, we discussed how finite bandwidth affects the surface coverage problem. In this section, we will discuss how finite band thickness and finite bandwidth affect the surface buildup problem.

After a layer of filament has been wound onto the surface, one will observe that the layer varies in thickness from point to point. This is due to the fact that the hoop angle cannot remain constant everywhere. One notices in particular that there is considerable buildup at and near the turning points. The objective of this section is to compute a uniform approximation to the layer thickness at all points of interest. The central problem of this section is determining the length of a cut, transverse to the axis of the surface, in the band of filaments. For a cylinder and at a point where the hoop angle is constant, the length of such a cut is given exactly by

$$c(x) = \frac{b}{\sin h}$$

where b is the bandwidth and h is the hoop angle. At the turning points, h is zero and the previous formula predicts infinite buildup. The buildup at the turning points may be considerable, but it is certainly not infinite. We will leave the determination of c for later and assume for the present that we will be able to obtain it.

We will first determine the physical surface length and the physical hub diameters for a given bandwidth b.

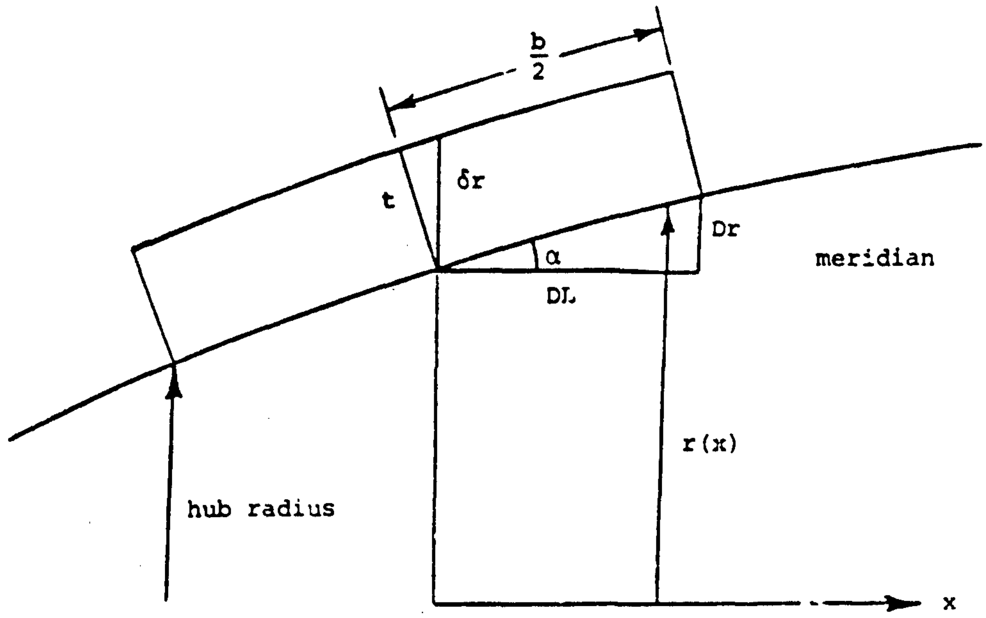

Figure 14. Left turning point.

$$\tan\alpha = r'(o) = \frac{Dr}{DL}$$

and

$$Dr^2 + DL^2 = \left(\frac{b}{2}\right)^2$$

$$(r'(o)DL)^2 + DL^2 = \left(\frac{b}{2}\right)^2$$

Hence

$$DL = \frac{b}{2\sqrt{1+r'(o)^2}}$$

and

$$Dr = \frac{br'(o)}{2\sqrt{1+r'(o)^2}}$$

The physical length of the surface is therefore given by

$$L + \frac{b}{2}\left(\frac{1}{\sqrt{1+r'(o)^2}} + \frac{1}{\sqrt{1+r'(L)^2}}\right)$$

and the left and right physical hub diameters are given by

$$2r(o) - \frac{br'(o)}{\sqrt{1+r'(o)^2}}$$

and

$$2r(L) + \frac{br'(L)}{\sqrt{1+r'(L)^2}}$$

respectively.

In what follows, we will let the δ quantities refer to a single band of filaments and the Δ quantities refer to the aggregation of these bands which form the layer.

From the same diagram we also have

$$t \simeq \delta r \cos \alpha$$

so that

$$\delta r \simeq t\sqrt{1+r'(x)^2}$$

for <u>any</u> point x and <u>any</u> orientation of the band.

We make the following assumption or idealization in our modeling of the behavior of the band of filaments. We assume that the band behaves as a flexible membrane of constant thickness t and width b which clings uniformly to the surface. The area of a transverse cut through the band is therefore given by

$$(\pi(r+\delta r)^2-\pi r^2)\ \frac{\delta\gamma}{2\pi} = \delta A$$

where $\delta\gamma$ is the angular length of the cut. Since $c = r\delta\gamma$, we have

$$\frac{c}{2r}\ (2r\delta r+\delta r^2) = \delta A$$

or

$$\delta A = c\delta r(1 + \frac{\delta r}{2r})$$

We will now have two δA area contributions from each circuit, hence, the total area contribution for our layers of wrap is given by

$$\Delta A = 2N_{CL}\delta A$$

Now, if we think of Δr as the average thickness of aggregate filament material buildup at any point, we have

$$\Delta A = \pi(r+\Delta r)^2 - \pi r^2$$

$$= \pi(2r\Delta r+\Delta r^2)$$

Solving this quadratic equation for Δr and rationalizing, we have

$$\Delta r = \frac{\Delta A/\pi}{r + \sqrt{r^2+\Delta A/\pi}}$$

Technically, the previous value of ΔA is due to filaments alone, while we should also consider the matrix material component. If we let ν be the filament/volume ratio, then ΔA for matrix and filament together is given by

$$\Delta A = 2N_{CL}\delta A/\nu$$

In summary, we have

$$\Delta r \simeq \frac{\Delta A/\pi}{r + \sqrt{r^2+\Delta A/\pi}}$$

$$\Delta A = 2N_{CL}\delta A/\nu$$

$$\delta A = c\delta r(1+\delta r/(2r))$$

$$\delta r = t\sqrt{1+r'^2}$$

The only quantity in these equations whose computation has not yet been addressed is the length of the transverse filamentary cut, c. We will now proceed to compute c. In order to complete our model of the band, we need to

define the family of paths orthogonal to our quasi-geodesic. These orthogonal paths will enable us to define precisely the finite width of our band and compute c.

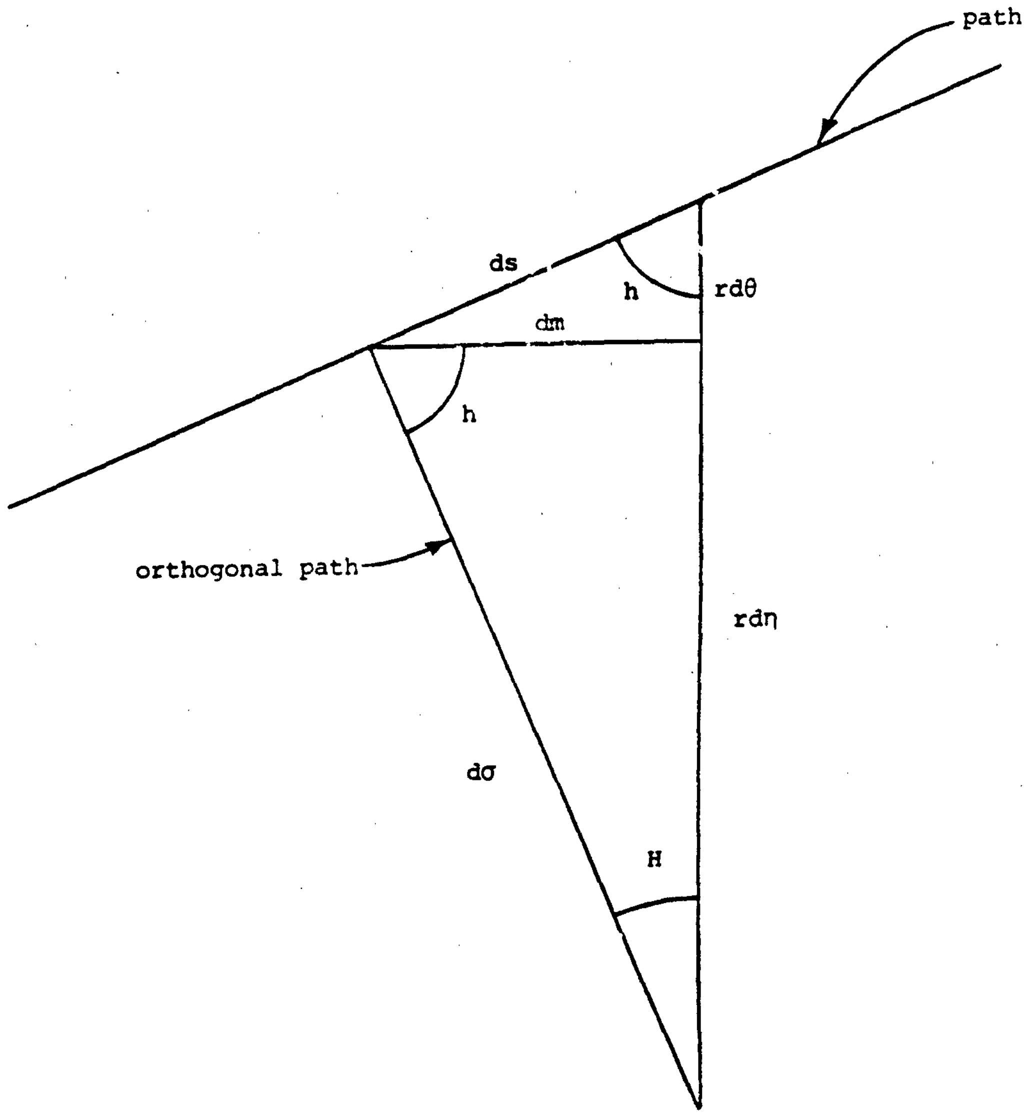

Figure 15. Path/orthogonal path relations.

We let s denote the length of our quasi-geodesic path, σ be the length of the corresponding orthogonal path, and m be the length of the meridian. The hoop angle H of the orthogonal path is defined by

$$\cos h = \sin H$$

The contribution to the meridian is

$$dm = d\sigma \sin H = d\sigma \cos h = \frac{r_o}{r} d\sigma$$

but

$$dm^2 = dx^2 + dr^2 = (1+r'^2)dx^2$$

Therefore

$$d\sigma = \frac{r}{r_o} dm = \frac{r}{r_o} \sqrt{1+r'^2}\, dx$$

hence

$$\sigma' = \frac{r}{r_o} \sqrt{1+r'^2}$$

We call the orthogonal path wrap angle η and we have

$$-rd\eta = d\sigma \cos H = d\sigma \sin h$$

$$= d\sigma(1-\cos^2 h)^{1/2}$$

$$= d\sigma(1-(r_o/r)^2)^{1/2}$$

$$= \frac{1}{r} d\sigma(r^2-r_o^2)^{1/2}$$

Hence

$$d\eta = -\frac{1}{r^2} (r^2-r_o^2)^{1/2} \cdot \frac{r}{r_o} (1+r'^2)^{1/2} dx$$

and

$$\eta' = -\frac{1}{rr_o} \{(r^2-r_o^2)(1+r'^2)\}^{1/2}$$

Note that η is a decreasing function of x.

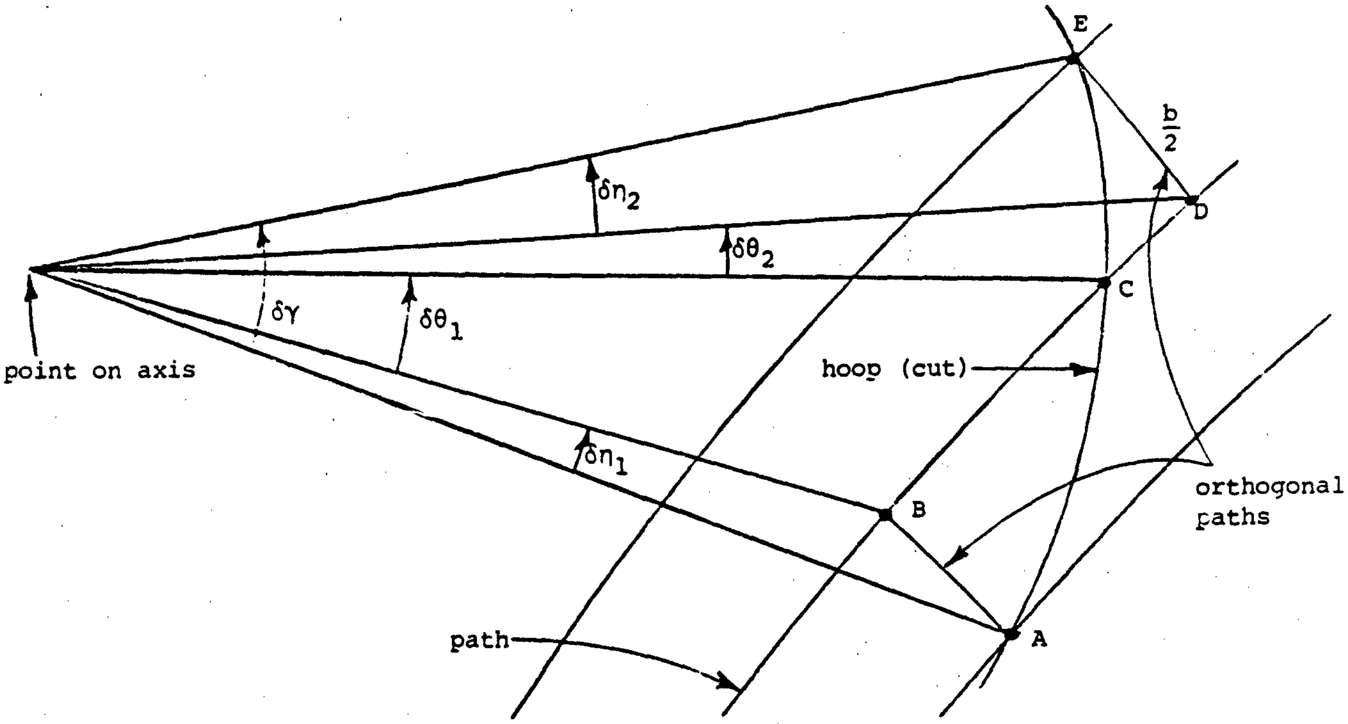

Figure 16. Determination of cut length.

Going from A to E indirectly by the route ABCDE, we pass through the wrap angle $\delta\gamma$, which is decomposed in terms of σ and s wrap angles:

$$\delta\gamma = \delta\eta_1 + \delta\theta_1 + \delta\theta_2 + \delta\eta_2$$

$$= \int_x^{x-\delta x_1} \eta'(t)dt + \int_{x-\delta x_1}^{x} \theta'(t)dt + \int_x^{x+\delta x_2} \theta'(t)dt + \int_{x+\delta x_2}^{x} \eta'(t)dt$$

$$= \int_{x-\delta x_1}^{x+\delta x_2} \theta'(t) - \eta'(t)dt$$

where x is the axial location of the cut and δx_1 and δx_2 are the axial distances therefrom to points B and D, respectively. Now

$$\theta' - \eta' = \frac{r_o}{r} \left(\frac{1+r'^2}{r^2-r_o^2}\right)^{1/2}$$

$$+ \frac{1}{rr_o} \{(r^2-r_o^2)(1+r'^2)\}^{1/2}$$

$$= \frac{1}{r} (1+r'^2)^{1/2}\left\{\frac{r_o}{(r^2-r_o^2)^{1/2}} + \frac{(r^2-r_o^2)^{1/2}}{r_o}\right\}$$

$$= \frac{1}{r}(1+r'^2)^{1/2}\frac{r_o^2 + r^2 - r_o^2}{r_o(r^2-r_o^2)^{1/2}}$$

$$= \frac{r}{r_o}\left(\frac{1+r'^2}{r^2-r_o^2}\right)^{1/2} = \gamma'$$

(note that the only difference between γ' and θ' is that θ' contains the factor r_o/r and γ' contains the factor r/r_o). The length of the transverse cut in the band is therefore given by

$$c(x) = r(x)\delta\gamma = r(x)\int_{x-\delta x_1}^{x+\delta x_2}\gamma'(t)dt$$

$$= r(x)(\gamma(x+\delta x_2)-\gamma(x-\delta x_1))$$

where

$$\gamma'(x) = \frac{r(x)}{r_o(x)}\left(\frac{1+r'(x)^2}{r(x)^2-r_o(x)^2}\right)^{1/2}$$

and δx_1 and δx_2 are determined by

$$\int_{x-\delta x_1}^{x}\sigma'(t)dt = \frac{b}{2} = \int_{x}^{x+\delta x_2}\sigma'(t)dt$$

where

$$\sigma'(x) = \frac{r(x)}{r_o(x)}(1+r'(x)^2)^{1/2}$$

In the special case when r and r_o are constant, we have as a check

$$\delta x_1\frac{r}{r_o} = \frac{b}{2} = \delta x_2\frac{r}{r_o}$$

$$\delta x_1 = \delta x_2 = \frac{br_o}{2r}$$

$$\gamma' = \frac{r}{r_o(r^2-r_c^2)^{1/2}} = \frac{1}{r_o(1-\cos^2 h)^{1/2}}$$

$$= \frac{1}{r_o\sin h} = \text{const}$$

$$c(x) = r \int_{x-\delta x_1}^{x+\delta x_2} \gamma'(t)dt = r \cdot 2\delta x_1 \cdot \gamma'$$

$$= 2r \cdot \frac{br_o}{2r} \cdot \frac{1}{r_o \sin h}$$

$$= \frac{b}{\sin h}$$

Since σ' is always greater than unity, σ is strictly monotone increasing and hence has an inverse σ^{-1}. We can therefore compute δx_1 and δx_2 in the following manner:

$$\sigma(x) - \sigma(x-\delta x_1) = \frac{b}{2}$$

$$\sigma(x-\delta x_1) = \sigma(x) - \frac{b}{2}$$

$$x - \delta x_1 = \sigma^{-1}(\sigma(x) - \frac{b}{2})$$

$$\delta x_1 = x - \sigma^{-1}(\sigma(x) - \frac{b}{2})$$

Also

$$\sigma(x+\delta x_2) - \sigma(x) = \frac{b}{2}$$

$$\sigma(x+\delta x_2) = \sigma(x) + \frac{b}{2}$$

$$x + \delta x_2 = \sigma^{-1}(\sigma(x) + \frac{b}{2})$$

$$\delta x_2 = \sigma^{-1}(\sigma(x) + \frac{b}{2}) - x$$

If we approximate σ' by a piecewise linear function, we have another application for our process of inverting the integral of a positive, continuous, piecewise linear function. We only need to decide what mesh to use for σ'.

Let hatted items denote piecewise linear estimates and unhatted items denote exact values. The error in computing δx_1 is therefore:

$$e(x) = \delta x_1 - \hat{\delta} x_1$$

$$= x - \sigma^{-1}(\sigma(x) - \frac{b}{2})$$

$$- (x - \hat{\sigma}^{-1}(\hat{\sigma}(x) - \frac{b}{2}))$$

$$= \hat{\sigma}^{-1}(\hat{\sigma}(x) - \frac{b}{2}) - \sigma^{-1}(\sigma(x) - \frac{b}{2})$$

Letting

$$\sigma^{-1} = S \quad \text{and} \quad \hat{\sigma}^{-1} = \hat{S}$$

we have

$$e(x) = \hat{S}(\hat{\sigma}(x) - \frac{b}{2}) - S(\sigma(x) - \frac{b}{2})$$

Assuming b to be small, we have

$$e(x) \approx \hat{S}(\hat{\sigma}(x)) - \frac{b}{2} \hat{S}'(\hat{\sigma}(x))$$

$$- S(\sigma(x)) + \frac{b}{2} S'(\sigma(x))$$

but

$$\hat{S}(\hat{\sigma}(x)) = x = S(\sigma(x))$$

hence

$$e(x) \approx \frac{b}{2} (S'(\sigma(x)) - \hat{S}'(\hat{\sigma}(x)))$$

Now, since

$$\sigma^{-1}(x) = S(x) \; ,$$

$$x = \sigma(S(x))$$

therefore

$$1 = \sigma'(S(x))S'(x)$$

or

$$S'(x) = 1/\sigma'(S(x))$$

Also

$$S'(\sigma(x)) = 1/\sigma'(S(\sigma(x))) = 1/\sigma'(x)$$

Similarly

$$\hat{S}'(\hat{\sigma}(x)) = 1/\hat{\sigma}'(x)$$

hence

$$e(x) \approx \frac{b}{2}\left(\frac{1}{\sigma'(x)} - \frac{1}{\hat{\sigma}'(x)}\right)$$

$$= \frac{b}{2}\,(\hat{\sigma}'(x) - \sigma'(x))/(\sigma'(x)\hat{\sigma}'(x))$$

A rough bound for e on the i^{th} subinterval is therefore

$$|e| \lesssim \frac{b}{2\sigma'^2} \cdot \frac{1}{8} h_i^2 \|\sigma'''\|_{(i)}$$

In order to make e roughly constant, we choose our g function as

$$g(x) = |\sigma'''(x)|^{\frac{1}{2}}/\sigma'(x)$$

Let us now obtain an asymptotic expression for the length of the cut at the left turning point (for one-half circuit). From geometric considerations we know that $\delta x_1 = 0$. We approximate δx_2 using a two-term Taylor series:

$$\delta x_2 = S(\sigma(x) + \frac{b}{2}) - x$$

$$\approx S(\sigma(x)) + S'(\sigma(x))\,\frac{b}{2} - x$$

$$= \frac{b}{2\sigma'(x)}$$

We use the previously derived quadrature rule to integrate γ':

$$\int_o^\delta \frac{f(x)}{\sqrt{x}}\,dx \approx 2\sqrt{\delta}\,f(\frac{\delta}{3})$$

$$c(o) = r(o)\int_o^{\delta x_2} \gamma'(x)dx = r(o)\int_o^{\delta x_2} \frac{\sqrt{x}\gamma'(x)}{\sqrt{x}}\,dx$$

$$\approx r(o) \cdot 2\sqrt{\delta x_2}\,\sqrt{\delta x_2/3}\;\gamma'(\delta x_2/3)$$

$$= \frac{2r(o)\delta x_2}{\sqrt{3}}\,\gamma'(\delta x_2/3)$$

but

$$\gamma'(x) = \frac{r(x)}{r_o(x)} \left(\frac{1+r'(x)^2}{r(x)^2-r_o(x)^2}\right)^{1/2}$$

Using a two-term Taylor series on r and r_o, we have

$$r(x) \sim r(o) + xr'(o)$$

$$r_o(x) \sim r(o) + xr_o'(o)$$

$$r(x)^2 - r_o(x)^2 \approx x(r'(o)-r_o'(o))(2r(o)+x(r'(o)+r_o'(o)))$$

$$\sim 2xr(o)(r'(o)-r_o'(c))$$

for small x. Therefore

$$\gamma'(x) \sim \left(\frac{1+r'(o)^2}{2xr(o)(r'(o)-r_o'(o))}\right)^{1/2}$$

for small x. Hence

$$c(o) = r(o)\int_o^{\delta x_2} \gamma'(x)dx \approx \frac{2r(o)\delta x_2}{\sqrt{3}} \left(\frac{1+r'(o)^2}{2r(o)\,\frac{\delta x_2}{3}\,(r'(o)-r_o'(o))}\right)^{1/2}$$

$$= \left(\frac{2r(o)\delta x_2(1+r'(o)^2)}{r'(o)-r_o'(o)}\right)^{1/2}$$

but

$$\delta x_2 = \frac{b}{2\sigma'(o)}$$

and

$$\sigma'(o) = (1 + r'(o)^2)^{1/2}$$

Therefore

$$\delta x_2 \approx \frac{b}{2(1+r'(o)^2)^{1/2}}$$

and

$$c(o) \approx \left(\frac{br(o)(1+r'(o)^2)^{1/2}}{r'(o)-r_o'(o)}\right)^{1/2}$$

If $r'(o)$ is large, we have

$$c(o) \approx (br(o))^{1/2}$$

If we use virtually the same analysis to approximate the cut at a distance of b/2 from the left turning point meridionally, we get

$$c(x) \sim \sqrt{2}\, c(o)$$

The length of the cut is therefore about 40 percent greater here than at the turning point.

FUNCTION DEFINITION

The off-line filament winding process begins with the explicit specification of the profile or radius function of the surface of revolution to be wound. We have been calling this function $r(x)$. Although one can, by exercising proper caution, wind across discontinuities in r' and r, we require here for the sake of simplicity that r' exist everywhere in $(0,L)$. This is not to say, however, that we cannot allow r' and r'' to be fairly large in some spots; we can.

The next step is to define the desired nominal path we wish the filament to follow on the surface. This is done by explicitly specifying the polar radius function $r_o(x)$. For any acceptable pure geodesic path, $r_o(x)$ is of course just a constant. For a quasi-geodesic path, however, we may need as much flexibility defining r_o as we need defining r.

The last function we need to specify, the perturbation function $p(x)$, is required for flexibility in specifying the slight modification which we must make in our nominal quasi-geodesic path in order to wrap the surface with a closed, uniformly covering layer of filament. In many cases, this function could simply be defined as a constant, but in the event that r' is large near the turning points, a constant p will have the undesirable effect of forcing the

effective polar radius function $\rho_0(x)$ to mimic the behavior of r near the turning points. Making p zero in a neighborhood of the turning points leaves ρ_0 equal to r_0 in these neighborhoods.

The smoothness of a function is specified in the following manner. A function f is said to be C^n if at least the first n derivatives of f exist and are continuous. The most stringent requirement on the smoothness needed by r, r_0, and p can be determined by examining the g function for generating the winder data. The extent of differentiation in this function indicates that r''', r_0'', and p'' should at least exist almost everywhere; r should be C^2 and r_0 and p should be C^1. Additionally, if we want g to be continuous, we will need r to be C^3 and r_0 and p to be C^2.

Of course it is not at all necessary for actual machining of the surface to be done this smoothly, but it is necessary for the functions we compute with to be this smooth in order to give us computationally reliable algorithms.

There are any number of ways in which we might define these functions. The method chosen here is to begin with a piecewise linear function and round off the corners to obtain the desired smoothness. We do the rounding off by using smoothing by averaging (refs 3,4). Consider the smoothing operator S defined by

$$Sf(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt \qquad h > 0$$

Differentiating this equation with respect to x using Leibnitz's rule gives us:

$$\frac{d}{dx} Sf(x) = \frac{1}{2h} (f(x+h)-f(x-h))$$

---

[3]H. S. Shapiro, Smoothing and Approximation of Functions, Van Nostrand Reinhold Company, New York, 1969.

[4]R. W. Soanes, "Function Smoothing by Repeated Averaging," ARDEC Technical Report ARCCB-TR-88012, Benet Laboratories, Watervliet, NY, March 1988.

and

$$\frac{d^{n+1}}{dx^{n+1}} Sf(x) = \frac{1}{2h} (f^{(n)}(x+h)-f^{(n)}(x-h))$$

Hence, we see that if f is C^n, then Sf is C^{n+1}. If we apply S i times, we also have that if f is C^n, then $S^i f$ is C^{n+i}.

We use the following symbolism for the successive integrals of a function f:

$$f_0(x) = f(x)$$

$$f_i(x) = \int_c^x f_{i-1}(t)dt \qquad i \geq 1$$

By definition of S,

$$Sf(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt = \frac{1}{2h} (f_1(x+h)-f_1(x-h))$$

$$S^2f(x) = \frac{1}{2h} \int_{x-h}^{x+h} Sf(t)dt$$

$$= \frac{1}{2h} \int_{x-h}^{x+h} \frac{1}{2h} (f_1(t+h)-f_1(t-h))dt$$

$$= \frac{1}{4h^2} (\int_{x-h}^{x+h} f_1(t+h)dt - \int_{x-h}^{x+h} f_1(t-h)dt)$$

$$= \frac{1}{4h^2} (\int_{x}^{x+2h} f_1(t)dt - \int_{x-2h}^{x} f_1(t)dt)$$

$$= \frac{1}{4h^2} (f_2(x+2h)-f_2(x)-(f_2(x)-f_2(x-h)))$$

$$= \frac{1}{4h^2} (f_2(x+2h)-2f_2(x)+f_2(x-2h))$$

In general, it can be proved (ref 4) that

$$S^i f(x) = \frac{1}{(2h)^i} \sum_{k=0}^{i} (-1)^k \binom{i}{k} f_i(x+(i-2k)h)$$

[4] R. W. Soanes, "Function Smoothing by Repeated Averaging," ARDEC Technical Report ARCCB-TR-88012, Benet Laboratories, Watervliet, NY, March 1988.

The j^{th} derivative of the i^{th} smooth is just

$$\frac{d^j}{dx^j} S^i f(x) = \frac{1}{(2h)^j} \sum_{k=0}^{i} (-1)^k \binom{i}{k} f_{i-j}(x+(i-2k)h)$$

If f is a bilinear function defined by

$$f(x) = \begin{cases} f(c) + a(x-c) & x > c \\ f(c) + b(x-c) & x < c \end{cases}$$

then we have almost immediately that

$$f_i(x) = \frac{(x-c)^i}{i!} \begin{cases} f(c) + \frac{a}{i+1}(x-c) & x > c \\ f(c) + \frac{b}{i+1}(x-c) & x < c \end{cases} \qquad (i \geq 0)$$

For the rounded area in the vicinity of x_k, we will take $c = x_k$. This establishes a,b and f_i for any value of i. In order to subsequently specify $S^i f$, we need to establish the one remaining parameter h. A point x will be called a linear point of $S^i f$ if x+ih and x-ih both lie in the domain of the same linear section of the initial piecewise linear approximation. We refer to such a point as linear because S^i preserves linear functions exactly, but since our basic approximation is only <u>piecewise</u> linear, S^i will only preserve the function at the point x if x+ih and x-ih both lie in the domain of the same linear section.

We want at least the midpoints of each subinterval to be linear points so that we may have different values of h associated with each node or rounded section. We therefore want the following two inequalities to be satisfied:

$$\frac{x_{k-1} + x_k}{2} + ih < x_k$$

and

$$\frac{x_k + x_{k+1}}{2} - ih > x_k$$

This leads us to define the h value associated with x_k to be

$$h = \frac{1}{2i} \min \{x_k - x_{k-1} \quad , \quad x_{k+1} - x_k\}$$

or some fraction thereof. Hence, $S^i f$ is defined between the midpoints of the subintervals surrounding x_k.

INDEFINITE INTEGRATION

In this section we define and analyze the complexity of an algorithm for indefinite numerical integration which tends to minimize the effects of round-off error in the limit as the number of mesh points becomes large. Typically, we want

$$F(x) = \int_{x_1}^{x} f(t)dt$$

estimated at the mesh points $x_1, x_2, \ldots x_n$, where n can be large. This can be accomplished by the usual recursion.

$$F_i = F_{i-1} + \int_{x_{i-1}}^{x_i} f(x)dx \qquad i = 2,3,\ldots,n$$

where $F_1 = 0$ and we estimate

$$\int_{x_{i-1}}^{x_i} f(x)dx$$

by any desired quadrature rule.

If n is large, the usual recursion will suffer from round-off error as we continually add relatively small quantities to increasingly larger ones. The larger n is and the greater the value of the accumulated integral, the greater will be the number of significant digits effectively dropped from each subinterval's contribution to the integral. It should be clear that F_n will suffer the most from round-off. To reduce the effects of round-off error, we construct an algorithm which will tend to add only quantities of similar magnitude and thereby make use of roughly the same number of significant digits in each addend. Consider the following algorithm:

Initiation step:

```
F(1):=0
i:=2
do until i > n
   {L(i):=i-1
    F(i):=∫_{x_{i-1}}^{x_i} f(x)dx
    i:=i+1}
```

Summation step:

```
do until L(n)=1
   {i:=n
    do until L(i)=1
       {k:=L(i)
        F(i):=F(i) + F(k)
        L(i):=L(k)
        i:=i-1}}
```

Examining the summation step, we see that during each pass of this step we have that F(i) and F(k) each represent integrals over the same number of subintervals, providing $L(k) \neq 1$. Hence, the new value of F(i) will be the sum of two numbers of similar magnitude (except for the last time F(i) is updated). We now examine the space-time complexity of the usual recursion and the new algorithm. The usual recursion needs an array of size n and the new algorithm needs two arrays of size n. Hence, both these algorithms have the same space complexity, O(n). The number of floating-point additions needed in the usual recursion is n-2; hence, the time complexity of the usual recursion is O(n). We will now analyze the time complexity of the new algorithm.

We first note how L changes from pass to pass via Table I (n=20).

TABLE I. BEHAVIOR OF L ARRAY FROM PASS TO PASS

x indices :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
L initially :		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
L after first pass :			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
L after second pass:				1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
L after third pass :						1	1	1	1	2	3	4	5	6	7	8	9	10	11	12
L after fourth pass:										1	1	1	1	1	1	1	1	2	3	4
L after fifth pass :																		1	1	1

On the basis of this table, we can construct the following table.

TABLE II. ADDITIONS PER PASS

p (pass index)	a_p (floating-point adds per pass)
1	n-2
2	n-2-1
3	n-2-1-2
4	n-2-1-2-4
5	n-2-1-2-4-8
.	.
.	.
.	.
p	$n-2-1-2-4-\ldots-2^{p-2}$

We therefore have that

$$a_p = n-2-\sum_{i=0}^{p-2} 2^i$$

by induction, but

$$S_k = \sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

Therefore

$$a_p = n-2-S_{p-2}$$
$$= n-2-(2^{p-1}-1)$$
$$= n-1-2^{p-1}$$

In all that follows, log will denote logarithms to the base 2 and ln will denote logarithms to the base e. For each pass, including the last, we have

$$a_p > 0$$
$$n-1-2^{p-1} > 0$$
$$2^{p-1} < n-1$$
$$p-1 < \log(n-1)$$
$$p < 1 + \log(n-1)$$

The cumulative number of adds in p passes is

$$A_p = \sum_{i=1}^{p} a_i = \sum_{i=1}^{p} (n-1-2^{i-1})$$

$$= p(n-1) - \sum_{i=1}^{p} 2^{i-1}$$

$$= p(n-1) - \sum_{i=0}^{p-1} 2^{i} = p(n-1) - S_{p-1}$$

$$= p(n-1) - (2^{p}-1)$$

The total number of floating-point additions in the summation step is therefore

$$A_p = p(n-1)-2^{p}+1$$

where p is the largest integer <u>strictly</u> less than

$$1 + \log(n-1)$$

If n-1 is an exact power of 2,

$$p = \log(n-1)$$

and

$$A_p = (n-1)\log(n-1) - (n-1) + 1$$
$$= (n-1)(\log(n-1)-\log 2) + 1$$
$$= (n-1)\log(\frac{n-1}{2}) + 1$$

If n-1 is not an exact power of 2,

$$p = 1 + \log(n-1) - \epsilon$$

where ϵ is some positive fraction. Hence

$$A_p = (n-1)(1+\log(n-1)-\epsilon)$$
$$- 2^{1+\log(n-1)-\epsilon} + 1$$
$$= (n-1)(1+\log(n-1)-\epsilon)$$
$$-2^{1-\epsilon}(n-1) + 1$$
$$= (n-1)(1+\log(n-1)-\epsilon-2^{1-\epsilon}) + 1$$

To get an upper bound on A_p, we need an upper bound on

$$f(\epsilon) = -\epsilon-2^{1-\epsilon}$$

Note that $f(0) = f(1) = -2$. Differentiating, we have

$$f'(\epsilon) = -1 - (-1)2^{1-\epsilon}\ln 2$$
$$= -1 + \frac{2^{1-\epsilon}}{\log e}$$

and

$$f''(\epsilon) = \frac{(-1)2^{1-\epsilon}\ln 2}{\log e} = \frac{-2^{1-\epsilon}}{(\log e)^2}$$

Since $f''(\epsilon) < 0$ for all ϵ, $f(\epsilon)$ attains a maximum. Setting $f'(\epsilon) = 0$ gives us

$$2^{1-\epsilon} = \log e$$
$$1-\epsilon = \log \log e$$
$$\epsilon = 1 - \log \log e$$

hence

$$f(\epsilon) = -1 + \log \log e - \log e$$
$$= -1 + \log \left(\frac{\log e}{e}\right)$$

Since

$$A_p = (n-1)(1+\log(n-1)+f(\epsilon)) + 1$$

for the correct ϵ, we have

$$A_p \leq (n-1)(1+\log(n-1)-1+\log(\frac{\log e}{e})) + 1$$

for the maximizing ϵ. Therefore,

$$A_p \leq (n-1)\log(\frac{n-1}{e \ln 2}) + 1$$

$$< (n-1)\log(\frac{n-1}{1.884}) + 1$$

which is our sought upper bound on the time complexity of the summation step.

We see that the price we have to pay for the reduction of round-off error is the logarithmic factor in our bound. Note, however, that this factor grows very slowly with n.

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